Modern Discrete Probability

An Essential Toolkit

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Modern Discrete Probability: An Essential Toolkit

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Preface

These lecture notes form the basis for a one-semester introduction to modern discrete probability with an emphasis on essential techniques used throughout the field. The material covered here is considered in much greater depth in the following excellent texts. (Consult the bibliography for complete citations.)

[AF] David Aldous and James Allen Fill. *Reversible Markov chains and random walks on graphs.*

[AS11] N. Alon and J.H. Spencer. *The Probabilistic Method*.

[BLM13] S. Boucheron, G. Lugosi, and P. Massart. *Concentration Inequalities: A Nonasymptotic Theory of Independence*.

[Gri10b] G.R. Grimmett. *Percolation*.

[JLR11] S. Janson, T. Luczak, and A. Rucinski. *Random Graphs*.

[LP] R. Lyons with Y. Peres. *Probability on trees and networks.*

[LPW06] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov chains and mixing times*.

[MU05] Michael Mitzenmacher and Eli Upfal. *Probability and Computing: Randomized Algorithms and Probabilistic Analysis*.

[RAS] Firas Rassoul-Agha and Timo Seppäläinen. *A course on large deviations with an introduction to Gibbs measures.*

[Ste] J. E. Steif. A mini course on percolation theory.

[vdH10] Remco van der Hofstad. Percolation and random graphs. In *New perspectives in stochastic geometry*.

[vdH14] Remco van der Hofstad. *Random graphs and complex networks.*

In fact, these notes are meant as a summary of some the basic topics covered in these more specialized references. My hope is that, by the end of the course, students will have picked up enough background to learn the advanced material on their own with greater ease. Many more sources were used in putting these notes together. They are acknowledged in the "Bibliographic remarks" at the end of each chapter. These notes were also strongly influenced by graduate courses of David Aldous, Elchanan Mossel, Yuval Peres, and Alistair Sinclair at UC Berkeley. Some of the material covered here can also be found in [Gri10a], with a different emphasis.

I assume that students have taken at least one semester of graduate probability at the level of [Dur10]. I am also particularly fond of [Wil91]. Some familiarity with countable Markov chain theory will be necessary, as covered for instance in [Dur10, Chapter 6]. An advanced undergraduate treatment such as [Dur12] will largely suffice however. See also [LPW06, Chapter 1].