WORKSHEET 3

① Find the following limits:

(a) $\lim_{x\to 1} \frac{x^2-1}{x^3-1}$ (b) $\lim_{x\to 2} (x^3+2x+1)$

(c) $\lim_{h\to 0} \frac{\sqrt{4+h}-2}{\sqrt{9+2h}-3}$ (d) $\lim_{t\to 2^+} \frac{\sqrt{t-2}}{t^2-4}$

(e) $\lim_{x\to\infty} \frac{-5x^2+1}{x^3+x+2}$ (f) $\lim_{x\to\infty} \frac{\sqrt{x^3+1}}{x^2}$

② Use the definition of the derivative and no other derivative rules to find $f'(x)$ if $f(x)=\frac{1}{4x^2+5}$

③ Solve the following equations:

(a) $e^{\ln(x^2+6)} - \ln(e^{5x}) = 0$ (b) $\ln(x^2+x) = \ln 6$

(c) $\ln(e^2) - e^{\frac{\ln(x)}{2}} = 0$ (d) $\ln\left(x+\frac{3}{2}\right) - \ln\left(\frac{1}{x}\right) = 0$

(e) $e^{2x} - 6e^x + 5 = 0$

④ Suppose that A and b are positive numbers with $\log_3 A = b$. Write the following logarithms as functions of b:

(a) $\log_3\left(\frac{3}{\sqrt{A}}\right)$ (b) $\log_{\frac{1}{3}} A$ (c) $\log_3(\sqrt{3A})$

(d) $\log_{\sqrt{3}} A$.

⑤ Let
$$f(x) = \begin{cases} x-1, & \text{if } x<0 \\ x+1, & \text{if } 0\le x\le 1 \\ 2x, & \text{if } 1<x\le 2 \\ x^2, & \text{if } 2<x \end{cases}$$

Where does $f(x)$ have discontinuities, if any?

(6) Find the constant c ~~such~~ so that the following function is continuous everywhere:

$$f(x) = \begin{cases} \dfrac{x^4-1}{x^3-1}, & \text{if } x \neq 1 \\ c, & \text{if } x = 1 \end{cases}$$

(7) Draw the graph of a function $f(x)$ which is defined on the interval $[-1, 5]$, is continuous everywhere, except at $x = 0, 1, 2, 3, 4$ and has no limit at $x = 0, 3$.

(8) Show that $\lim\limits_{x \to 1} \dfrac{x-1}{|x-1|}$ does not exist.

(9) In 1950, carbon dating was used to determine the age of wood samples excavated from a city in ancient Babylon. The rate of radioactive decay of these samples was measured at 4.09 disintegrations per minute (dpm). By comparison, the decay rate from fresh wood samples was measured at 6.68 dpm. Using 5,568 years as the half-life of radioactive carbon, estimate the age of the samples.

(10) Consider the function $y = x^3 + x$.

(a) Find the equation of the tangent line at $(-1, 0)$.

(b) Where is the tangent line parallel to the line $2y - 26x + 4 = 0$?

(c) Is the tangent line ever horizontal for any point on the graph? Explain.

(11) A souvenir company sells 10,000 Beanie Babies per day and is operating at a net profit of \$5200 per day. Its marginal revenue is \$1.40 and its marginal cost is \$1.25. Estimate how much its profit per day will increase if it increases sales by 200 Beanie Babies per day.

(12) Use linear approximation to estimate $\frac{1}{2.108}$

(13) Find all points on the graph of $y = \ln\left(\frac{2}{x^3}\right) + \frac{x^2}{2} - 2x$ where the tangent line is horizontal.

(14) Consider a curve defined by the following equation:

$$(x+1)^2 + y^2 + \ln(x+1) = (1 - \ln y)\, e^{x^2} + y$$

(a) Find the slope of the tangent line to this curve as a function of x and y.

(b) Find the equation of the tangent line at $(0,1)$.

(15) Use implicit differentiation to find $\frac{dy}{dx}$ if

$$x e^{y^2} + 14xy = 0.$$

(16) The volume of a sphere of radius r is given by the formula $V = \frac{4}{3}\pi r^3$. When helium is pumped into a spherical balloon, both the radius and the volume change with respect to time. Suppose that at a certain moment the volume is increasing at the rate of 8π cubic feet per minute and the radius is increasing at the rate of $\frac{1}{2}$ foot per minute. What is the volume of the balloon at that moment?

(17) The Incredible Hulk is standing on top of a building 12 meters tall and pulling on a rope that is attached to a car. As he pulls on the rope, the car is being pulled toward the building. Suppose that the Hulk is able to pull the rope at a rate of 10 meters per minute. How fast is the car moving toward the building when it is 5 meters away?

(18) Suppose the demand for a certain product is given by $q = 5000\, e^{-0.08p}$, where p is the price and q is the number sold at that price.

(a) What is the rate of change of demand with respect to price when the price is 10?

(b) Find the relation between the rate of change of demand and the rate of change of price when the price is 10.

(c) Assume that the price depends on time according to the following equation:

$$p^3 + e^{-0.1t}\, p = 2.$$

What is the rate of change of demand when $t = 0$ and $p = ?$ (Hint: use implicit differentiation to get $\frac{dp}{dt}$).

(19) Consider the function $f(x) = \frac{-x^2}{x+1}$, $x \neq -1$.
Find where the function is increasing / decreasing.
Find the critical points and the local max / min points.
Does global max / min exist?

(20) Consider the function $f(x) = e^{x^2}$. Determine where f is concave up and concave down. Find the inflection points.

(21) Use the 2nd derivative test to find local maxima / minima of the function $f(x) = xe^{-x^2}$.

(22) Let $f(x) = x^5 + x + 7$.

(a) Check that $f(-2) < 0$ and $f(-1) > 0$. Explain why the equation $x^5 + x + 7 = 0$ must have a solution between $x = -2$ and $x = -1$.

(b) Show that $f(x)$ has no critical points. Explain why the equation $x^5 + x + 7 = 0$ cannot have two solutions. (Hint: what is the sign of $f'(x)$ and what does it tell you?)

(23) Show that the equation $e^x - x = 0$ does not have a solution. (Hint: Find the critical point and show that it is a global minimum. Then check its value).

(24) Suppose that f is a function with domain $(-\infty, \infty)$ such that $f'(x) < 0$ for $x < 1$, $f'(x) > 0$ for $x > 1$, $f(1) = 2$ and $f'(1) = 0$. How many solutions does the equation $f(x) = 0$ have?

(25) Sketch the graphs of the following functions:

(a) $f(x) = \dfrac{2x}{x^2-4}$ (b) $f(x) = \dfrac{1}{x^2+4}$

(c) $f(x) = x + e^{-x}$ (d) $f(x) = x - \dfrac{4}{x}$

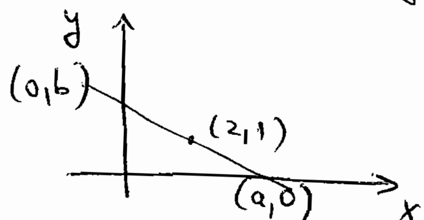
(Note: Make sure you indicate any local min/max points and inflection points, as well as any asymptotes).

(26) Find the global max and or / min (if any) of the following functions on the given interval:

(a) $f(x) = \dfrac{e^x}{x^2}$ on $(0, \infty)$ (b) $f(x) = \dfrac{x^3}{3} - 2x^2 + 3x - 1$ on $[0, 2]$.

(27) Find the point on the line $y = x$ that is closest to the point $(0, 2)$. What is the minimum distance?

(28) Find the line through the point $(2,1)$ that cuts off the triangle of smallest area in the first quadrant (Hint: Let t be the slope of the line and find the area of the triangle as a function of t).



(29) The price per unit of a certain item is a function of the number of units produced and is given by

$$p = \frac{1000}{250 + x} - 1, \quad 0 < x < 750.$$

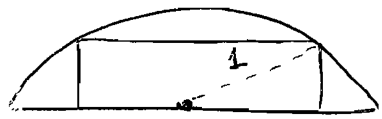
Find the x that maximizes the revenue.

(30) You want to build a box with square base and top to have a surface area of 600 square inches. What is the maximum possible volume?

(31) Suppose you want to enclose 600 square yards with a rectangular fence and then divide the area in half with a fence parallel to one of the sides. What dimensions require the least amount of fencing?

(32) Find the area of the largest rectangle that can be inscribed in a semicircle of radius 1 with the base of the square along a diameter of the

circle, as shown below.



(33) Find the following indefinite integrals:

(a) $\displaystyle\int \frac{e^{2x}}{e^x+1}\,dx$ (b) $\displaystyle\int \frac{x}{x+2}\,dx$ (c) $\displaystyle\int \frac{2x+1}{(3x^2+3x+1)^3}\,dx$

(d) $\displaystyle\int e^{\sqrt{x}}\,dx$ (e) $\displaystyle\int x^5 e^{x^2}\,dx$ (f) $\displaystyle\int \frac{1-e^x}{1+e^x}\,dx$

(g) $\displaystyle\int (\ln x)^3\,dx$ (h) $\displaystyle\int \frac{e^x}{e^{2x}-4e^x+3}\,dx$ (i) $\displaystyle\int \frac{1}{(\sqrt{x}-2)(\sqrt{x}+3)}\,dx$

(34) Solve the initial value problem

$$\frac{dy}{dx} = \frac{2\ln x}{x}, \quad y(1) = -1$$

(35) Solve the initial value problem

$$\frac{dy}{dt} = e^{2t}\sqrt{e^t+2}, \quad y(0) = 1$$

(36) Evaluate the following definite integrals:

(a) $\displaystyle\int_0^1 \frac{6}{t^2-t-2}\,dt$ (b) $\displaystyle\int_0^4 \frac{x}{1+\sqrt{x}}\,dx$

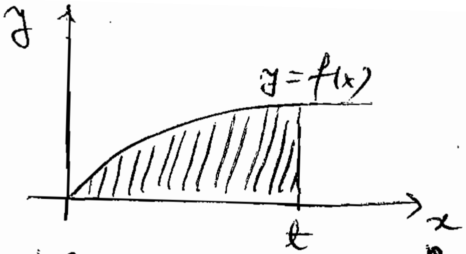
(37) Sketch the graph of the function and use formulas from elementary geometry (area of a rectangle and area of triangle) to find the definite integral

$\displaystyle\int_0^3 (1-x)\,dx.$

(38) Find both $\int_a^b f(x)\,dx$ and the total area between the graph of f and the x-axis from a to b for

$f(x) = x^3$, $a = -1$, $b = 2$.

(39) In the graph shown below, the shaded area is equal to $t + e^{-t} - 1$ for any $t > 0$. What is $f(x)$?

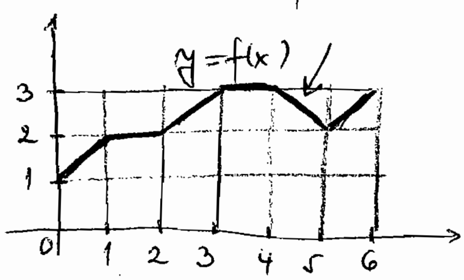


(40) If $G(t) = 5t^2 + \int_{\sqrt{t}}^{1} e^x \sqrt{1+x^3}\,dx$, compute $G'(1)$.

(41) Find the area enclosed by the curves $y = x^2$ and $y = -x^2+8$

(42) Find the area between the graph of $f(x) = \frac{8}{x^2}$ and $g(x) = x$ over the interval $1 \le x \le 3$.

(43) Find the average value of the function $f(x) = xe^{x/3}$, $0 \le x \le 3$

(44) 

Suppose f is the function whose graph is shown to the right.

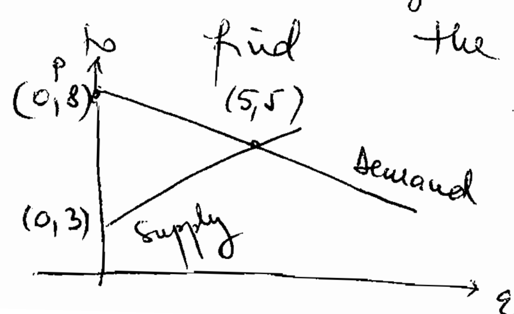
Find (a) $\int_0^5 f(x)\,dx$

(b) $\int_0^3 f(x)\,dx$

(c) $\int_3^0 f(x)\,dx$

(45) A company operating at the production level of 30 units per day estimates that its marginal cost is given by the function $C'(x) = x^2 - 10x + 80$ dollars per unit. Find the additional cost for operating at the level of 40 units per day.

(46) The figure below shows supply and demand curves, both of which are straight lines. Use elementary geometry to find the consumer and producer surplus.



(47) Find the equilibrium quantity and price and the CS and PS for $D(q) = \frac{110}{q+4}$, $S(q) = q+5$

(48) A self-employed software engineer estimates that her annual income over the next 10 years will steadily increase according to the formula $70{,}000(1.2)^t$, where t is the time in years. She decides to save 12% of her income in an account paying 6% annual interest, compounded continuously. Treating the savings as a continuous income stream over a 10-year period, find the present value.

(49) An employee of a company is offered a choice between two retirement plans. In the 1st plan the company will deposit an initial amount of \$5,000 in an account paying interest at an annual rate of 8%, compounded continuously and it will then continue to deposit money at a constant rate of \$15,000 per year for the next 25 years. In the 2nd plan, the company will pay the employee \$1,100,000 at the end of 25 years. Which one is the more beneficial plan for the employee?

(50) Solve the initial value problem

$$\frac{dy}{dt} = y^2 + y, \quad y(0) = 1$$

(51) A person opens a retirement account with an initial amount of \$2,000. After that, money is continuously deposited in the account at the rate of \$9,000 per year. Assume that the interest rate is 9%, compounded continuously. Model this problem as a differential equation and and initial condition describing the amount of money, $M(t)$, in the account at any time t. Then solve it to find $M(t)$ at any time t and the balance after 25 years.

(52) A home buyer can afford to spend no more than \$900 per month on mortgage payments. Suppose that the annual interest rate is 8%, compounded continuously, that the term of the mortgage is 30 years and that payments are also made continuously.

(a) Determine the maximum amount that this buyer can afford to borrow

(b) Determine the total interest paid during the term of the mortgage.

(53) Solve the logistic equation $\frac{dp}{dt} = p - p^2$ with $p(0) = 5$ and compute $\lim_{t\to\infty} p(t)$.

(54) Sketch the level curves in the xy-plane corresponding to the given heights z_0. Then describe the x- and y- sections for each function where x and y are fixed at the given values.

(a) $f(x,y) = 1 + x + y$; $z_0 = -3, -2, -1, 0, 1, 2$; $x = 0, y = -1$

(b) $f(x,y) = (x-1)^2 + y^2$; $z_0 = 0, \frac{1}{4}, 1, 4, 9$; $x = 2, y = 1$

(55) Find the equation of the plane passing through the points $(1,2,1)$, $(-1,-3,-6)$, $(2,8,-2)$.

(56) Find the first order partial derivatives of the given functions

(a) $f(x,y) = y^3 e^{2xy} - x^5$ (b) $f(x,y) = e^{-xy^2} \ln(x^2+1)$

(57) Find the equation of the tangent plane to the graph of the given function at the given point and then find the linear approximation to the function at the same point:

(a) $f(x,y) = \ln(x^2 + 3y^2 + 2)$, $(0,0)$

(b) $f(x,y) = x^2 e^{yx} + y^3$, $(0,-2)$

(58) The distance from the origin to a point (x,y) in the plane is $\sqrt{x^2+y^2}$. Use linear approximation to estimate the change in distance from the origin in moving from the point $(3,4)$ to the point $(3.012, 4)$.

(59) Find the critical points of the following functions and determine their nature (max / min / saddle point):

(a) $f(x,y) = 2x^2 - x^4 - y^4$ (b) $f(x,y) = y + \frac{1}{2}x^2 - \ln(xy^2)$

(60) A firm produces two kinds of products: X, which sells for \$3 each and Y, which sells for \$5 each. The company determines that the total cost, in thousands of dollars, of producing x ~~thou~~ thousand of X and y thousand of Y is given by $C(x,y) = 2x^2 - 2xy + y^2 - 3x + y + 7$. Find the amount of each type of product that must be produced and sold in order to maximize profit.

GOOD LUCK

ON THE FINAL!