WHAT YOU NEED TO KNOW ABOUT SERIES

Remember how we defined an infinite sum:

$$\sum_{k=0}^{\infty} a_k = \lim_{N\to\infty} \sum_{k=0}^{N} a_k = \lim_{N\to\infty} S_N.$$

Whenever you deal with an infinite series, there are two sequences lurking in the background. These are the sequence of *partial sums*: $\{S_1, S_2, S_3, \dots\}$ and the sequence of *terms*: $\{a_1, a_2, a_3, \dots\}$. The 'no way' or 'n^{th}-term test is originally formulated by stating that if a series $\sum_{k=1}^{\infty}$ converges, then $\lim_{k\to\infty} a_k = 0$. The useful application of this is the contra-positive of this statement. That is, if $\lim_{k\to\infty} a_k \neq 0$, then the series $\sum_{k=1}^{\infty}$ diverges. This is more commonly known as the n^{th}-term test. DO NOT say that $\lim_{n\to\infty} a_n = 0$ implies $\sum_{k=1}^{\infty} a_k$ converges. I believe this is one of the most common 222 mistakes regarding sequences and series.

You should know the difference between the two sequences S_n and a_k, and how to go back and forth between them. Most of the theorems we never covered discuss statements about necessary and sufficient conditions regarding the behavior of the a_k for a series to converge.

You need to know the geometric series: $1 + r + r^2 + \cdots = \frac{1}{1-r}$ provided $|r| < 1$. If $|r| \geq 1$, the series diverges. I find writing the terms out easier than juggling with the Σ notation: $\sum_{k=\text{whatever}}^{\infty} ar^k$.

You need to know about the telescoping series. These are series where one may compute the partial sums S_N directly, then take the limit of these. As always, *if the limit of the partial sums does not exist, the series diverges!* Two examples to think about are

$$\sum_{k=1}^{\infty} \left(\sqrt{k+1} - \sqrt{k}\right) \qquad \text{and} \qquad \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n}\right).$$

I believe this is basically all you need to know about sequences and series.