

Circle One:

Name: _____

7:45-8:35 (361)

8:50-9:40 (362)

Math222-4, Spring 2007

Quiz #3 (Take Home): 02-16-07

Due: 02-19-07

You may discuss this quiz solely with me or other students in my discussion sessions only. Use a new sheet of paper for each problem. Write on one side only. Illegible solutions will not be graded. Do not cite an integral table as justification for your answer. All of these problems can be done using the standard methods used for homework problems.

1. (2 Points) Compute:

$$\int \frac{1}{x^{7/6} + x} dx.$$

Hint: $x^{7/6} + x = x(x^{1/6} + 1)$.

Solution: We begin by factoring the denominator, and substituting $u = x^{1/6}$. Hence, $du = 1/6x^{-5/6}dx$, so $dx = 6x^{5/6} du = 6u^5 du$.

$$I = \int \frac{1}{x^{7/6} + x} dx = \int \frac{1}{x(x^{1/6} + 1)} dx = \int \frac{6u^5}{u^6(u + 1)} du = \int \frac{6}{u(u + 1)} du.$$

The integral involving u 's can be solved using partial fractions.

$$\begin{aligned} I &= 6 \int \frac{1}{u} + \frac{-1}{u + 1} du = 6(\ln |u| - \ln |u + 1|) + C = 6 \ln |u/(u + 1)| + C \\ &= 6 \ln |x^{1/6}/(x^{1/6} + 1)| + C. \end{aligned}$$

2. (2 Points) Integrate:

$$\int \frac{d\theta}{\sqrt{1 + \sqrt{\theta}}}.$$

Solution: We'd like to get rid of the ugliest part of this integral, the $\sqrt{\theta}$ part. Let's try substituting $x = \sqrt{\theta}$. Then $dx = 1/(2\sqrt{\theta}) d\theta$, so $d\theta = 2x dx$. We have:

$$I = \int \frac{d\theta}{\sqrt{1 + \sqrt{\theta}}} = \int \frac{2x}{\sqrt{1 + x}} dx = 2 \int \frac{x}{\sqrt{x + 1}} dx.$$

Now this integral looks slightly simpler. Let's sub in for the part inside the square root sign. Put $u = x + 1$, so that $du = dx$ and $x = u - 1$.

$$\begin{aligned} I &= 2 \int \frac{u - 1}{\sqrt{u}} du = 2 \int (u^{1/2} - u^{-1/2}) du = 2(2/3u^{3/2} - 2u^{1/2}) + C \\ &= 4(1/3(x + 1)^{3/2} - (x + 1)^{1/2}) + C = 4(1/3(x + 1)^{3/2} - (x + 1)^{1/2}) + C \\ &= 4(1/3(\sqrt{\theta} + 1)^{3/2} - (\sqrt{\theta} + 1)^{1/2}) + C. \end{aligned}$$

3. (2 Points) Compute:

$$\int_{-1}^3 \frac{4x^2 - 7}{2x + 3} dx.$$

Solution: You should immediately recognize that this problem can be solved using partial fractions. Secondly, over the interval $[-1, 3]$ the denominator is never 0, so we can proceed as usual. We need to do long division:

$$\begin{array}{r} 2x - 3 \\ 2x + 3 \overline{) 4x^2 - 7} \\ \underline{- 4x^2 - 6x} \\ - 6x - 7 \\ \underline{6x + 9} \\ 2 \end{array}$$

This gives us

$$\begin{aligned} I &= \int_{-1}^3 \frac{4x^2 - 7}{2x + 3} dx = \int_{-1}^3 \left(2x - 3 + \frac{2}{2x + 3}\right) dx = \int_{-1}^3 \left(2x - 3 + \frac{1}{x + 3/2}\right) dx \\ &= (x^2 - 3x + \ln|x + 3/2|) \Big|_{-1}^3 = (9 - 9 + \ln|3 + 3/2|) - (1 + 3 + \ln|1 + 3/2|) \\ &= \ln|9/2| - 4 - \ln|5/2| = \ln|9/5| - 4. \end{aligned}$$

4. (2 Points) Integrate:

$$\int \frac{1}{(r + 1)\sqrt{r^2 + 2r}} dr.$$

Solution: First, let's complete the square on the inside of the radical. We know $r^2 + 2r = (r + 1)^2 - 1$, so it is tempting to make a sub for this $r + 1$. Let $x = r + 1$, then $dx = dr$. The integral becomes:

$$I = \int \frac{1}{(r + 1)\sqrt{r^2 + 2r}} dr = \int \frac{1}{x\sqrt{x^2 - 1}} dx.$$

We should recognize that this can be solved using a trig-substitution. Put $x = \sec(\theta)$, so that $dx = \sec(\theta)\tan(\theta) d\theta$. Then,

$$\begin{aligned} I &= \int \frac{\sec(\theta)\tan(\theta) d\theta}{\sec(\theta)\sqrt{\sec^2(\theta) - 1}} = \int \frac{\tan(\theta)}{\sqrt{\tan^2(\theta)}} d\theta = \int d\theta = \theta + C = \sec^{-1}(x) + C \\ &= \sec^{-1}(r + 1) + C. \end{aligned}$$

5. (2 Points) Compute:

$$\int_0^{\ln(2)} \frac{e^t}{e^{2t} + 3e^t + 2} dt.$$

Solution: Substitute $u = e^t$, $du = e^t dt$. The denominator factors becomes $u^2 + 3u + 2 = (u + 2)(u + 1)$. At $t = 0$, $u = 1$ and at $t = \ln(2)$, $u = 2$, so we have

$$I = \int_1^2 \frac{1}{(u + 1)(u + 2)} du.$$

The problem is now reduced to a partial fractions problem. We put

$$\frac{1}{(u+1)(u+2)} = \frac{A}{u+1} + \frac{B}{u+2}$$

and solving for the coefficients using your favorite method, we have $A = 1$ and $B = -1$. Therefore,

$$I = \int_1^2 \left(\frac{1}{u+1} + \frac{-1}{u+2} \right) du = \ln |(u+1)/(u+2)| \Big|_1^2 = \ln(3/4) - \ln(2/3).$$

6. (2 Points) The region in the first quadrant that is enclosed by the x -axis, the curve $y = 5/x\sqrt{5-x}$, and the lines $x = 1$ and $x = 4$ is revolved about the x -axis to generate a solid. Find the volume of the solid.

Solution: To find the volume, we need use the washer method and compute

$$V = \pi \int_1^4 y^2 dx = \pi \int_1^4 \frac{25}{x^2(5-x)} dx.$$

This integral can be solved using partial fractions:

$$\frac{25}{x^2(5-x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(5-x)}$$

Multiplying through by the denominator gives: $25 = Ax(5-x) + B(5-x) + Cx^2$. Plugging in $x = 0$ and $x = 5$ we get B and C for free: $B = 5$ and $C = 1$. This reduces the equation to $25 = Ax(5-x) + 5(5-x) + x^2$. Plugging in $x = 1$, we have $25 = 4A + 20 + 1$, so that $A = 1$ as well. We have,

$$\begin{aligned} V &= \pi \int_1^4 \left(\frac{1}{x} + \frac{5}{x^2} + \frac{1}{(5-x)} \right) dx = \ln|x| - 5/x - \ln|5-x| \Big|_1^4 \\ &= \ln(4) - 5/4 - \ln(1) - \ln(1) + 5 + \ln(4) = 2\ln(4) + 15/4. \end{aligned}$$

7. (2 Points) Suppose $n, m \in \{1, 2, 3, \dots\}$, the set of positive integers. Calculate $\frac{1}{\pi} \int_0^{2\pi} \sin(mx) \sin(nx) dx$. *Hint:* Treat the case $n = m$ as a special case.

Solution:

For the case $n = m$, this problem reduces to finding

$$1/\pi \int_0^{2\pi} \sin^2(nx) dx = 1/(2\pi) \int_0^{2\pi} (1 - \cos(2nx)) dx = 1/(2\pi) (x - 1/(2n) \sin(2nx)) \Big|_0^{2\pi}.$$

Since for any integer n , we have $\sin(4\pi n) = \sin(0) = 0$, we have $1/\pi \int \sin^2(nx) = 1$.

In the case $n \neq m$ we use the angle sum formulas for $\sin(x)$. We have

$$\begin{aligned} \int_0^{2\pi} \sin(mx) \sin(nx) dx &= \int_0^{2\pi} \frac{\cos((m-n)x) - \cos((m+n)x)}{2} dx \\ &= 1/2 \left(\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n} \right) \Big|_0^{2\pi} = 0 \end{aligned}$$

since both $\sin(0) = \sin(\text{integer} \cdot 2\pi) = 0$. Note that it is important to split this problem up into two parts. The angle sum formula *is* valid when $n = m$, but when we did the integration for the term involving $\cos((n - m)x)$, we had to divide by $n - m$. In fact, when $n = m$, we precisely recover the formula used when we integrated $\sin^2(nx)$.

8. (2 Points) Suppose $\int_{-\infty}^{\infty} f(x) dx = 1$ where $f(x) = C|x|e^{-kx^2}$ for some $k > 0$. *Hint:* For each of these parts, calculate $\int_0^{\infty} f(x) dx$ and $\int_{-\infty}^0 f(x) dx$ separately. Add your answers.

(a) Find C .

(b) Find the value $\mu := \int_{-\infty}^{\infty} xf(x) dx$.

Solution: We seek $\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx = I + J$ where I is the integral over the negative part and J is the integral over the positive part.

We first compute $I = \int_0^{\infty} C|x|e^{-kx^2} dx$. For positive x , $|x| = x$, so we can drop the absolute value signs. Let $u = kx^2$, then $du = 2kx dx$. We have

$$\begin{aligned} I &= \int_0^{\infty} Cxe^{-kx^2} dx = C \lim_{R \rightarrow \infty} \int_{x=0}^{x=R} xe^{-u} du / (2kx) = C/(2k) \lim_{R \rightarrow \infty} \int_{x=0}^{x=R} e^{-u} du \\ &= -\frac{C}{2k} \lim_{R \rightarrow \infty} e^{-u} \Big|_{x=0}^{x=R} = -\frac{C}{2k} \lim_{R \rightarrow \infty} e^{-kx} \Big|_{x=0}^{x=R} = -\frac{C}{2k} \lim_{R \rightarrow \infty} (e^{-Rk} - 1) = \frac{C}{2k}. \end{aligned}$$

The term $e^{-kR} \rightarrow 0$ as $R \rightarrow \infty$ since $k > 0$.

For negative x , we have that $|x| = -x$. Then

$$J = \int_{-\infty}^0 |x|e^{-kx^2} dx = \lim_{R \rightarrow \infty} \int_{x=-R}^{x=0} (-x)e^{-kx^2} dx.$$

If we substitute $y = -x$, we actually have

$$J = \lim_{R \rightarrow \infty} \int_{y=R}^{y=0} -ye^{-ky^2} dy = \lim_{R \rightarrow \infty} \int_{y=0}^{y=R} ye^{-ky^2} dy = I.$$

Therefore

$$1 = \int_{-\infty}^{\infty} f(x) dx = I + J = 2I = \frac{C}{k},$$

so $C = k$.

For part (b), you really need to calculate these integrals separately. Again, we split up $\int_{-\infty}^{\infty} = \int_{-\infty}^0 + \int_0^{\infty}$.

$$I = \int_0^{\infty} xf(x) dx = \lim_{R \rightarrow \infty} \int_0^R x|kx|e^{-kx^2} = \lim_{R \rightarrow \infty} \int_0^R kx^2e^{-kx^2} dx.$$

This can be solved using integration by parts:

$$\begin{aligned} u &= kx & du &= kdx \\ dv &= xe^{-kx^2} dx & v &= -1/(2k)e^{-kx^2}. \end{aligned}$$

$$\begin{aligned}
I &= \lim_{R \rightarrow \infty} \left(\frac{-xe^{-kx^2}}{2} \Big|_{x=0}^{x=R} + 1/2 \lim_{R \rightarrow \infty} \int_0^R e^{-kx^2} \right) \\
&= \lim_{R \rightarrow \infty} \frac{-Re^{-kR^2}}{2} + 1/2 \lim_{R \rightarrow \infty} \int_0^R e^{-kx^2} dx = \frac{1}{2\sqrt{k}} \lim_{R \rightarrow \infty} \int_0^{\sqrt{k}R} e^{-y^2} dy.
\end{aligned}$$

The first term vanishes if you apply L'Hospital's rule one time:

$$\lim_{R \rightarrow \infty} \frac{-R}{e^{kR^2}} \stackrel{L.H.}{=} \lim_{R \rightarrow \infty} \frac{-1}{2kRe^{kR^2}} = 0.$$

The second term is evaluated using the substitution $y = x\sqrt{k}$. We have

$$I = 1/(2\sqrt{k}) \int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{4\sqrt{k}}.$$

If we compute $J = \int_\infty^0 xf(x) dx$ using the same exact methods, we get $J = -\frac{\sqrt{\pi}}{4\sqrt{k}}$, so the total integral is $\int_{-\infty}^\infty xf(x) dx = 0$.

9. (4 Points) The **Laplace Transform** is defined as $\mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-st} dt$. It is common to write $\hat{f}(s) := \mathcal{L}\{f(t)\}(s)$.

(a) If $f(t) = t$, find $\hat{f}(s)$, where $s > 0$.

(b) If $f(t) = e^{\alpha t}$, find $\hat{f}(s)$, where $s > \alpha$.

(c) If $f(t) = \sin(\alpha t)$, find $\hat{f}(s)$, where $s > 0$.

(d) If $f(t) = \cos(\alpha t)$, find $\hat{f}(s)$, where $s > 0$.

Why must we assume these restrictions on s ?

Solution (a): We need to compute $\int_0^\infty te^{-st} dt$. This can be done by parts:

$$\begin{aligned}
u &= t & du &= dt \\
dv &= e^{-st} dt & v &= -\frac{e^{-st}}{s}.
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty te^{(-st)} dt &= \lim_{R \rightarrow \infty} \int_0^R te^{(-st)} dt = \lim_{R \rightarrow \infty} \left(\frac{-t}{se^{st}} \Big|_{t=0}^{t=R} + \int_0^R e^{-st}/s dt \right) \\
&= \lim_{R \rightarrow \infty} \left(\frac{-R}{se^{sR}} - \frac{1}{s^2} e^{-st} \Big|_{t=0}^{t=R} \right) = \lim_{R \rightarrow \infty} \left(\frac{-R}{se^{sR}} - \frac{-R}{s^2 e^{sR}} \right) + \frac{1}{s^2} \\
&= 1/s^2.
\end{aligned}$$

All the terms involving an R go to zero. You can verify this using L'Hospital's rule.

Solution (b):

$$\begin{aligned}
\int_0^\infty e^{\alpha t} e^{-st} dt &= \lim_{R \rightarrow \infty} \int_0^R e^{(\alpha-s)t} dt = \lim_{R \rightarrow \infty} \frac{e^{(\alpha-s)t}}{\alpha-s} \Big|_{t=0}^{t=R} \\
&= 1/(\alpha-s) \lim_{R \rightarrow \infty} (e^{(\alpha-s)R} - 1) = 1/(s-\alpha).
\end{aligned}$$

Since $\alpha - s < 0$, the term involving R goes to zero.

Solution (c): Using integration by parts twice, one can verify that

$$\int e^{-st} \sin(\alpha t) dt = \frac{e^{-st}}{s^2 + \alpha^2} (-s \sin(\alpha t) - \alpha \cos(\alpha t)) + C.$$

To do this, start with picking $u = e^{-st}$, and $dv = \sin(\alpha t) dt$. The integral that gets pushed out can be integrated again by parts, this time setting $u = e^{-st}$, and $dv = \cos(\alpha t) dt$. You'll recover the original integral the second time. Add it to both sides and divide by the number that's produced.

We have,

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_0^R e^{-st} \sin(\alpha t) dt &= \lim_{R \rightarrow \infty} \frac{e^{-st}}{s^2 + \alpha^2} (-s \sin(\alpha t) - \alpha \cos(\alpha t)) \Big|_{t=0}^{t=R} \\ &= \lim_{R \rightarrow \infty} \frac{e^{-sR}}{s^2 + \alpha^2} (-s \sin(\alpha R) - \alpha \cos(\alpha R)) + \frac{\alpha}{s^2 + \alpha^2} \\ &= \frac{\alpha}{s^2 + \alpha^2}. \end{aligned}$$

Solution (d): This problem is essentially the same. as the previous one. First, you need to show

$$\int e^{-st} \cos(\alpha t) dt = \frac{e^{-st}}{s^2 + \alpha^2} (-s \cos(\alpha t) + \alpha \sin(\alpha t)) + C.$$

Use the same exact methods you used to do part (c).

We have,

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_0^R e^{-st} \cos(\alpha t) dt &= \lim_{R \rightarrow \infty} \frac{e^{-st}}{s^2 + \alpha^2} (-s \cos(\alpha t) + \alpha \sin(\alpha t)) \Big|_{t=0}^{t=R} \\ &= \lim_{R \rightarrow \infty} \frac{e^{-sR}}{s^2 + \alpha^2} (-s \cos(\alpha R) + \alpha \sin(\alpha R)) + \frac{s}{s^2 + \alpha^2} \\ &= \frac{s}{s^2 + \alpha^2}. \end{aligned}$$

We need to make the assumptions we did on s otherwise the integrals wouldn't converge.