Math 101 Review of SOME Topics

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1 BASICS

1.1 Fractions

I know you all learned all this years ago, but I will still go over it...

Take a fraction, say $\frac{5}{7}$. You can interpret this as follows: you cut a cake in seven (equal, needless to say) pieces and take 5 of those pieces. Then $\frac{10}{7}$ means you have 10 of those pieces. Obviously, seven of those pieces will give you a whole cake and you will have 3 such pieces left; thus $\frac{10}{7}$ is equal to $1\frac{3}{7}$. Another way to see that is

$$\frac{10}{7} = \frac{7+3}{7} = \frac{7}{7} + \frac{3}{7} = 1 + \frac{3}{7}$$

A very common practice that we employ when we want to add two fractions is to change the way they look so that they have the same denominator. Let's talk about changing the way a fraction looks...Going back to $\frac{5}{7}$ example...remember we divided the cake into 7 pieces, and took 5 of them. Now if we divide every piece into 3 equal pieces, in total you will get 21 pieces and the ones that you took will give you 15 pieces. Hence

$$\frac{5}{7} = \frac{5 \cdot 3}{7 \cdot 3} = \frac{15}{21}$$

What we have just done is called "changing the denominator" of a fraction but be careful, while we are changing the bottom, we are also changing the top. Infact, we are multiplying BOTH the top and the bottom by the SAME nonzero number!

1.1.1 Studied Examples and Problems

1. (Addition) $\frac{5}{7} + \frac{3}{4}$

   We have 5 pieces of size one seventh and we have 3 pieces of size one fourth. Since sizes of the pieces are different, we cannot go ahead and say we have 8 pieces. We need to adjust the sizes of the pieces, in other words, we need to put the fractions in COMMON DENOMINATOR:

the denominators are 4 and 7. A common multiple (try to use the least common multiple in general) for these two denominators is 28. So

$$\frac{5}{7}+\frac{3}{4}=\frac{(5)(4)}{(7)(4)}+\frac{(3)(7)}{(4)(7)}=\frac{20}{28}+\frac{21}{28}$$

Now that they have the same denominator (thus each piece has the same size), we can combine the two fractions:

$$\frac{41}{28}$$

2. (Multiplication) $\frac{5}{7}\cdot\frac{3}{4}$
   Multiplication is straightforward: we multiply the tops and the bottoms

$$\frac{5}{7}\cdot\frac{3}{4}=\frac{5\cdot 3}{7\cdot 4}=\frac{15}{28}$$

3. **Assignment** Do as indicated: $\frac{4}{6}+\frac{3}{8}$, $\frac{2}{15}+\frac{4}{5}$, $\frac{3}{4}\frac{8}{13}$, $\frac{5}{14}+\frac{5}{6}$
4. (**Be Careful**) $\frac{2x+y}{x}\neq 2+y$
   What should happen is the following:

$$\frac{2x+y}{x}=\frac{2x}{x}+\frac{y}{x}=2+\frac{y}{x}$$

1.2 Order of Operations

So it goes as follows: paranthesis, power/root, multiplication/division, addition/subtraction

As you see, multiplication and division are in the same rank. For these guys, we work in order, from left to right. Same for addition and subtraction.

1.2.1 Studied Examples and Problems

1. $24\div 3\cdot 2+6$

   First multiplication and division, from left to right, then addition. Thus the order is

$$((24\div 3)\cdot 2)+6=22$$

2. Here's an example lots of people did incorrectly in the third exam:

   All these people worked out the problem until the end and got $3-36\sqrt[3]{3}$. This was the answer but they went ahead and did the substitution first and got $-33\sqrt[3]{3}$. This is WRONG because multiplication comes before subtraction, we have to multiply 36 and $\sqrt[3]{3}$ first, which gives

$36\sqrt[3]{3}$ and then subtract that from 3 which gives $3 - 36\sqrt[3]{3}$. Thus there's no simplification whatsoever.

$$3 - 36\sqrt[3]{3} \neq -33\sqrt[3]{3}$$

3. $-2^2 \neq (-2)^2$

   Another common mistake, in the left hand side (abbreviated as LHS in what follows) we square first and then multiply by -1, thus LHS is -4. In the RHS (you know what this stands for), we have paranthesis first, and then power, thus we take the square of -2, which is 4.

2 Roots and Powers

Given a real number a and a positive integer n, we define

$$a^n := a \cdot a \cdots a \cdot a \qquad (n \ \ times)$$

Then we extend this definition to negative integers as

$$a^{-n} := \frac{1}{a^n}$$

Now let's talk about roots: Given a positive real number a and a positive even integer n, there are two numbers ,one opposite of the other, whose nth power is a. These two numbers are called the nth roots of a, the positive one is called the *principal* nth root of a. The principal one is denoted with $\sqrt[n]{a}$ and then, the negative one becomes $-\sqrt[n]{a}$.

For example, there are two square roots of 9, one is positive, other is negative, namely 3 and -3. The positive one is the principal one and thus we denote them as

$$3 = \sqrt{9} \quad and \quad -3 = -\sqrt{9}$$

When n is odd, there is only one nth root of a.

Let's get back to exponentiation. Let's extend exponentiation to rational exponents:

$$a^{\frac{1}{n}} := \sqrt[n]{a}$$

This basically enables us to pass between exponent and radicals:

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

2.1 Main Rules

1. $a^n a^m = a^{n+m}$
2. $\dfrac{a^n}{a^m} = a^{n-m}$
3. $(ab)^n = a^n b^n$
4. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
5. $(a^m)^n = a^{mn} = (a^n)^m$ (Double Exponentiation Rule)

6. $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$
7. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

2.2 Be Careful

The nth root of a negative number is *not* real only if n is even. Example:

$$\sqrt{-8}\ is\ not\ real\ but\ \sqrt[3]{-8} = -2\ is\ real$$

Taking nth root and taking nth power are opposite of each other, thus one would "cancel" the other. But be careful when the index n is even:

$$(\sqrt[n]{a})^n = \sqrt[n]{a^n} = |n|\ if\ n\ is\ even$$

Example:

$$\sqrt[100]{(-3)^{100}} = |-3| = 3 \quad BUT \quad \sqrt[101]{(-3)^{101}} = -3$$

2.3 Studied Examples and Problems

1. (Simplifying Expressions with Exponents)
   (a) $\left(\dfrac{b^{-3/2}}{c^{-5/3}}\right)^2 (b^{-1/4}c^{-1/3})^{-1}$

   We start with the outmost exponents, distributing them over all the factors inside the paranthesis using rules number 3 and 4 above.

   $$\frac{(b^{-3/2})^2}{(c^{-5/3})^2} \cdot (b^{-1/4})^{-1} \cdot (c^{-1/3})^{-1}$$

   Now we use the double exponentiation rule (rule number 5 above)

$$\frac{b^{-6/2}}{c^{-10/3}} \cdot b^{1/4} \cdot c^{1/3}$$

$$\frac{b^{-6/2} \cdot b^{1/4} \cdot c^{1/3}}{c^{-10/3}}$$

Now we use rule number 1 on the b's and rule number 2 on the c's:

$$b^{(-6/2)+(1/4)} \cdot c^{(1/3)-(-10/3)}$$

$$b^{-11/4} \cdot c^{11/3}$$

(b) **Assignment** Simplify: $\frac{(a^2b^5)^{-1/4}}{(a^{-3}b^2)^{1/6}}$, $\frac{4m^{5/3}(m^{-2/3} - 4m^{-5/3})}{m^{-4/3}}$

2. (Simplifying Radicals, numbers inside)

   (a) $\sqrt{80}$

   Since we are taking "square" root, we need to the find biggest factor of 80 that is a "perfect square". If you make the tree diagram that we saw in class, you will see that $80 = 2^4 \cdot 5$, thus the biggest perfect square factor of 80 is 16.

   $$\sqrt{80} = \sqrt{16 \cdot 5} = 4\sqrt{5}$$

   (b) $\sqrt[3]{80}$

   This time we need to find the biggest "perfect cube" factor of 80 because we are taking "cubic" root. It is 8.

   $$\sqrt[3]{80} = \sqrt[3]{8 \cdot 10} = 2\sqrt[3]{10}$$

   (c) $\sqrt[4]{80}$

   This time we need to find the biggest "perfect fourth power" factor of 80 because we are taking "fourth" root. It is 16.

   $$\sqrt[4]{80} = \sqrt[4]{16 \cdot 5} = 2\sqrt[4]{5}$$

   (d) $\sqrt[5]{80}$

   This time we need to find the biggest "perfect fifth power" factor of 80 because we are taking "fifth" root. From the tree diagram factorization, we know that $80 = 2^4 \cdot 5$. Thus, there's no perfect fifth power factor of 80.This means we cannot simplify $\sqrt[5]{80}$.

   (e) (**Assignment**) Simplify the followings: $\sqrt{40}, \sqrt[3]{40}$, $\sqrt[4]{96}$, $\sqrt[5]{96}$, $\sqrt[3]{81}$, $\sqrt[4]{81}$

3. (Simplifying Radicals, variables inside)

   (a) $\sqrt{x^{10}}$ As in the previous case, we look for the biggest perfect square factor of x^{10}. It is x^{10} itself because $x^{10} = (x^5)^2$. So

   $$\sqrt{x^{10}} = x^5$$

   Actually, we have an easy alternative approach in this case that I want you to learn and use. Let me explain:

   Remember our connection between exponents and radicals:

   $$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

   If you apply this to $\sqrt{x^{10}}$, you get $x^{\frac{10}{2}} = x^5$. See next example.

   (b) $\sqrt[3]{x^{10}}$

   Following our last approach, this time we get $x^{\frac{10}{3}}$. Since 3 doesn't go into 10, $x^{\frac{10}{3}}$ is not really a "simplification". So 10 didn't work with 3, we go down by one and try 9 instead of 10. Obviously 3 goes into 9, so we should proceed as follows:

   $$\sqrt[3]{x^{10}} = \sqrt[3]{x^9 \cdot x} = x^{\frac{9}{3}}\ \sqrt[3]{x} = x^3\ \sqrt[3]{x}$$

   (c) $\sqrt[4]{x^{10}}$

   This time we get $x^{\frac{10}{4}}$. Since 4 doesn't go into 10, $x^{\frac{10}{4}}$ is not really a "simplification". So 10 didn't work with 4, we go down by one and try 9 instead of 10. Since 4 doesn't go into 9, so we should go down one more and try 8. Clearly 4 goes into 8, so it goes as follows :

   $$\sqrt[4]{x^{10}} = \sqrt[4]{x^8 \cdot x^2} = x^{\frac{8}{4}}\ \sqrt[4]{x^2} = x^2\ \sqrt[4]{x^2}$$

   (d) (**Assignment**) Simplify the followings: $\sqrt{x^9}, \sqrt[3]{y^{21}}$, $\sqrt[4]{y^{21}}$, $\sqrt[5]{x^{100}}$, $\sqrt[3]{x^{101}}$

4. (Simplifying Radicals, general case)

   (a) $\sqrt[3]{24 \cdot x^{22} \cdot y^{14}}$

   Greatest perfect cube factor of 24 is 8. We should take the x^{21} part of x^{22} and y^{12} part of y^{14}. So it goes like this:

$$\sqrt[3]{24x^{22}y^{14}} = \sqrt[3]{8 \cdot 3 \cdot x^{21} \cdot x \cdot y^{12} \cdot y^2} = 2x^{\frac{21}{3}}y^{\frac{12}{3}}\sqrt[3]{3xy^2} = 2x^7y^4\sqrt[3]{3xy^2}$$

   (b) (**Assignment**) Simplify: $\sqrt{28x^9}, \sqrt[3]{54y^{21}x^{29}}$, $\sqrt[4]{72y^{21}z^9}$, $\sqrt[5]{128x^{100}t^{36}}$

5. (Simplifying Expressions with Radicals)

   (a) $9\sqrt[3]{5q} - 2q\sqrt[3]{40q^4}$

   First observe that

$$\sqrt[3]{40q^4} = \sqrt[3]{8 \cdot 5q^3q} = 2q^{\frac{3}{3}}\sqrt[3]{5q} = 2q\sqrt[3]{5q}$$

   Now putting the two terms together:

$$9\sqrt[3]{5q} - 2q\sqrt[3]{40q^4} = 9\sqrt[3]{5q} - 2q(2q\sqrt[3]{5q}) = 9\sqrt[3]{5q} - 4q^2\sqrt[3]{5q}$$

   Observe that both terms have the factor $\sqrt[3]{5q}$, let's pull it out:

$$9\sqrt[3]{5q} - 4q^2\sqrt[3]{5q} = (\sqrt[3]{5q})(9 - 4q^2)$$

   (b) (**Assignment**) Simplify: $\sqrt[3]{64xy^2} + \sqrt[3]{27x^4y^5}$, $2\sqrt[4]{m^9p^6} - 3m^2p\sqrt[4]{mp^2}$

6. (Rationalizing the Denominator) Sometimes we can get rid of those ugly radicals in the denominator. Look at the examples:

   (a) $\dfrac{2-\sqrt{3}}{\sqrt{5}}$

   Remember that multiplying BOTH the top and the bottom of a fraction by the same nonzero number will give an equivalent fraction. And also remember that

$$(\sqrt{a})^2 = (\sqrt{a})(\sqrt{a}) = a$$

   So we should multiply both the top and the bottom of $\dfrac{2-\sqrt{3}}{\sqrt{5}}$ by $\sqrt{5}$.

$$\frac{2-\sqrt{3}}{\sqrt{5}} = \frac{2-\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{(2-\sqrt{3})(\sqrt{5})}{\sqrt{5}} = \frac{2\sqrt{5}-\sqrt{3}\sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{2\sqrt{5}-\sqrt{15}}{5}$$

(b) $\dfrac{3-\sqrt[3]{2}}{\sqrt[3]{5}}$

This time we will use the fact that

$$(\sqrt[3]{a})^3 = (\sqrt[3]{a})(\sqrt[3]{a})^2 = a$$

It goes like this:

$$\frac{3-\sqrt[3]{2}}{\sqrt[3]{5}} = \frac{3-\sqrt[3]{2}}{\sqrt[3]{5}} \cdot \frac{(\sqrt[3]{5})^2}{(\sqrt[3]{5})^2} = \frac{3(\sqrt[3]{5})^2 - \sqrt[3]{2}(\sqrt[3]{5})^2}{\sqrt[3]{5}(\sqrt[3]{5})^2} = \frac{3\sqrt[3]{25} - \sqrt[3]{2}\sqrt[3]{25}}{\sqrt[3]{5}(\sqrt[3]{5})^2} = \frac{3\sqrt[3]{25} - \sqrt[3]{50}}{5}$$

(c) $\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{5}}$

Here is an important technique: when we have the sum or the difference of two SQUARE root radicals, we use the CONJUGATE of the denominator to do the rationalizing. Given the sum or the difference of two square root radicals, to get its radical, we simply switch the sign: the conjugate of $\sqrt{2}-\sqrt{5}$ is $\sqrt{2}+\sqrt{5}$. Now, as usual we multiply BOTH the top and bottom:

$$\frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{5}} = \frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{5}} \cdot \frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}+\sqrt{5}} = \frac{(2-\sqrt{3})(\sqrt{2}+\sqrt{5})}{(\sqrt{2}-\sqrt{5})(\sqrt{2}+\sqrt{5})}$$

Now if you FOIL the bottom, you will see that it will become an integer. Actually, this was the whole purpose of using the conjugate. Maybe you have recognized, the bottom is the right side of the "Difference of the Two Squares" identity:

$$A^2 - B^2 = (A-B)(A+B)$$

So FOIL'ing is a loss of time at this point, we use the above identity instead:

$$\frac{(2-\sqrt{3})(\sqrt{2}+\sqrt{5})}{(\sqrt{2}-\sqrt{5})(\sqrt{2}+\sqrt{5})} = \frac{(2-\sqrt{3})(\sqrt{2}+\sqrt{5})}{(\sqrt{2})^2-(\sqrt{5})^2}$$

$$= \frac{2\sqrt{2}+2\sqrt{5}-\sqrt{3}\sqrt{2}-\sqrt{3}\sqrt{5}}{2-5} = \frac{2\sqrt{2}+2\sqrt{5}-\sqrt{6}-\sqrt{15}}{-3}$$

We are done.

(d) (**Assignment**) Rationalize the denominator: $\dfrac{\sqrt{2}}{\sqrt{3}}$, $\dfrac{\sqrt[3]{2}}{\sqrt[3]{3}}$, $\dfrac{\sqrt[4]{2}}{\sqrt[4]{32}}$ (hint: first simplify the radical) , $\dfrac{\sqrt{3}}{\sqrt{2}+\sqrt{3}}$, $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

3 Polynomials and Factorization

A polynomial is a term or a finite sum of terms in which all variables have whole number exponents. We will concentrate mostly on polynomials with one variable, sometimes two variables.

In particular, a polynomial in a variable t is a sum of the form

$$a_n t^n + a_{n-1} t^{n-1} + \ldots + a_2 + t^2 + a_1 t + a_0$$

To each such polynomial, we assign a number called the *degree* of the polynomial, to show how "big" the polynomial is. The degree of a polynomial (in one variable) is the biggest power of the variable that's present in the polynomial. So while $x^{11} - \frac{1}{2}x$ is a polynomial of degree 11 in x, 5 is a polynomial of degree 0 (in what variable ?)

We can add/subtract and multiply/divide polynomials. We will give examples in the Problems section. Another important aspect is to $factorize$ polynomials. One can think of factorization as the opposite of multiplication. Given two polynomials, we get a "bigger" polynomial by multiplying them. Factorization writes a "big" polynomial as a product "smaller" polynomials. These small polynomials are called $factors$ of the big guy.

example: multiply three degree one polynomials, we get a degree three polynomial

$$(x)(x-1)(x+1) = (x)(x^2-1) = x^3 - x$$

Now factorize the degree three polynomial to get its smaller factors back:

$$x^3 - x = x(x^2-1) = (x)(x-1)(x+1)$$

3.1 Some VERY Useful Identities

1. (**Difference of Two Squares**) $A^2 - B^2 = (A-B)(A+B)$
2. (Square of a Sum) $(A+B)^2 = A^2 + 2AB + B^2$
3. (Square of a Difference) $(A-B)^2 = A^2 - 2AB + B^2$

As you can see now

$$(x+y)^2 \neq x^2 + y^2$$

3.2 Studied Examples and Problems

1. (pulling out the Greatest Common Factor)
   (a) $28x^3y^3 - 42x^4y^2z$

   We look at the common factors that both terms have. Let's start with the numbers: 28 and 42, the greatest common factor of these guys is 14. Now look at the variables; the most x we can take out from both terms is x^3 and the most y we can take out is y^2. We cannot take any z out since the first term has no z. Thus we have

$$28x^3y^3 - 42x^4y^2z = 14x^3y^2(2y - 3xz)$$

(b) **Assignment** Factor $8x^3 - 4x, \quad 15x^3y^2 - 10x^2y^{5/2} + 5xyz$

2. (grouping) When there is no common factor among the terms of the expression, we may want to group terms that have common factos together:

$$6ax + 12bx + a + 2b$$

We have 4 terms with no common factor. But we may group terms **with** common factors together; there maybe a few grouping options, they should all work, don't worry, just go with one. In the above example, one can go with grouping the first two and the last two together:

$$6ax + 12bx + a + 2b = (6ax + 12bx) + (a + 2b) = 6x(a + 2b) + (a + 2b) = (6x + 1)(a + 2b)$$

We could have grouped differently, like this for example:

$$6ax + 12bx + a + 2b = (6ax + a) + (12bx + 2b) = a(6x + 1) + 2b(6x + 1) = (a + 2b)(6x + 1)$$

Assignment Factor $p^2q^2 - 10 - 2q^2 + 5p^2, \quad m^3 + 4m^2 - 6m - 24$

3. (the $x^2 + bx + c$ case)

Say we are given a polynomial of the form $x^2 + bx + c$. What we want to do is to factor it into two linear polynomail. On other words, we are looking for two numbers A and B such that

$$x^2 + bx + c = (x + A)(x + B)$$

So how will we find these A and B? Let's FOIL the RHS and see what we really have on RHS:

$$x^2 + bx + c = x^2 + (A + B)x + AB$$

So we see that in order to have RHS equal to LHS, we need to have the coefficient of x and the constant terms on both sides to be equal:

$$b = A + B \quad \textit{and} \quad c = AB$$

We are very close now, with a little inspection we can see if the desired A and B exist. Start with factors of c FIRST, then see if you can find a pair whose sum is b. Look at the following examples:

(a) $x^2 - 8x + 15$

We are looking for A and B such that $AB = 15$ and $A + B = -8$. We start with examining factors of 15: 5 and 3, -5 and -3, 15 and 1, -15 and -1. From these pairs, the one we want is -5 and -3 as we want their sum to be -8. So

$$x^2 - 8x + 15 = (x - 3)(x - 5)$$

(b) $3x^3 - 3x^2 - 36x$

Well, this is not the type of polynomial we are studying in this section. But once you take out that common factor $3x$ from all three terms, we will be fine:

$$3x^3 - 3x^2 - 36x = (3x)(x^2 - x - 12)$$

So let's find A and B such that $AB = -12$ and $A + B = -1$. Since the product is negative, one should be positive and the other should be negative. Also looking at their sum, we see that their absolute values should be close. By inspection we see that what we need is the pair -4 and 3. So

$$x^2 - x - 12 = (x - 4)(x + 3)$$

Thus

$$3x^2 - 3x - 36 = (3x)(x^2 - x - 12) = (3x)(x - 4)(x + 3)$$

(c) (NOTE) Sometimes we do not have the desired A and B, for example try $x^2 + x + 1$, you will not succeed.

(d) **Assignment** Factor $x^2 - 4x - 12, \quad k^2 - 11k + 30, \quad 4p^3 + 24p^2 - 64p$

4 Solving Equations

An equation is basically two mathematical expressions set equal to each other. Given an equation, $A = B$, there are two way we can manipulate it to put it into different looking but EQUIVALENT form:

1. (ADD/SUBTRACT) If we add or subtract a real number C to BOTH sides of the original equality, we get an equivalent equation:

$$A = B$$

is equivalent to

$$A + C = B + C$$

So this means we can "move" terms from one side to the other with the cost of SWITCHING its sign:

example:

$$4x - 5 = 3x - 7$$

Add 5 to both sides

$$4x - 5 + 5 = 3x - 7 + 5$$

$$4x = 3x - 7 + 5$$

Now subtract $3x$ from both sides

$$4x - 3x = 3x - 7 + 5 - 3x$$

Hence we get

$$4x - 3x = -7 + 5$$

So you see, we "moved" the -5 from left hand side to right hand side as $+5$ and we "moved" the $3x$ from right to left as $-3x$. Ideally you should use this "moving" shortcut!

2. (MULTIPLY/DIVIDE) If we multiply or divide BOTH sides of the original equality by a real number $C \neq 0$, we get an equivalent equation:

$$A = B$$

is equivalent to

$$A \cdot C = B \cdot C$$

and to

$$\frac{A}{C} = \frac{B}{C}$$

This enables us to the following: Say you have

$$5x = 10$$

To leave x alone, we go ahead and divide both sides by 5

$$\frac{5x}{5} = \frac{10}{5}$$

and get

$$x = 2$$

4.1 Linear Equations

A linear equation (in one variable) is an equation of the form

$$Ax + B = 0$$

For example, $3x - 5 = 4 - 5x$ is a linear equation although at first it looks different than the above form. Well, after we move everything to one side we get $8x - 9 = 0$, which fits our definition. So basically, all we want is that the equation should involve only one variable and the exponents should all be 1.

To solve such an equation, we basically isolate the variable. Here, let's work on an example:

$$3x - 5 - 2(3x - 4) = 4 - 5x + 7 - 8(x - 2)$$

First, we do the necessary multiplications

$$3x - 5 - 6x + 8 = 4 - 5x + 7 - 8x + 16$$

and then collect all the terms that have the variable on one side and all the numbers on the other side. REMEMBER, when you move a term to the other side, you SWITCH its sign:

$$3x - 6x + 5x + 8x = 4 + 7 + 16 + 5 - 8$$

Now tidy up both sides:

$$10x = 24$$

Make a division to leave x alone now:

$$x = \frac{24}{10} = \frac{12}{5}$$

4.1.1 Studied Examples and Problems

The above example is enough for this topic, so i'll just go ahead and assign some problems:

Assignment Solve: $7x - 3(4 - 2x) = 5 + 6x - 3$, $4t - 8 = 5(3 - 4(2t + 2))$

4.2 Solving Equations By Factorization

We will recall a very important property: ZERO FACTOR PROPERTY . It basically says that product of nonzero numbers is always nonzero. In other words, if a product is zero then at least one factor must be zero.

$$if \quad A \cdot B = 0 \quad then \quad A = 0 \quad or \quad B = 0$$

4.2.1 Studied Examples and Problems

1. $-x^3 + x^2 = -6x$ We move everything to one side first:

   $$0 = x^3 - x^2 - 6x$$

   Next step is to factor the RHS: we take out the common factor x:

   $$0 = (x)(x^2 - x - 6)$$

   Now let's factor the second factor on RHS; we look for A and B such that $AB = -6$ and $A + B = -1$. The pair we need is -3 and 2:

   $$x^2 - x - 6 = (x - 3)(x + 2)$$

   Thus we get

   $$0 = x^3 - x^2 - 6 = (x)(x^2 - x - 6) = (x)(x - 3)(x + 2)$$

   Use the Zero Factor Property now,

   $$x = 0 \quad or \quad x - 3 = 0 \quad or \quad x + 2 = 0$$

   Thus

   $$x = 0 \quad or \quad x = 3 \quad or \quad x = -2$$

2. $5z^2 = 25z$

   Again we move everything to one side and we factor:

   $$5z^2 - 25z = 0$$

   $$(5z)(z - 5) = 0$$

By Zero Factor Property

$$5z = 0 \quad or \quad z - 5 = 0$$

$$z = 0 \quad or \quad z = 5$$

3. **Assignment** Solve $48 + 3x^2 - x = 4x^2 + 7x, \quad 6t^3 + 5t^2 = 6t + 5$

4.3 Quadratic Equations

A quadratic equation (in one variable) is an equation of the following form:

$$ax^2 + bx + c = 0$$

Looking at its *discriminant* ($= b^2 - 4ac$), we can see what kind of solutions we have. There are 3 cases:

1. $b^2 - 4ac < 0$ Then our equation has no real solutions (it has "imaginary" solutions)
2. $b^2 - 4ac = 0$ Then our equation has only one solution and it is real (it is actually a rational number)
3. $b^2 - 4ac > 0$ Then our equation has two real solutions

To find the solutions we use the QUADRATIC ROOT FORMULA:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is a very powerful formula, sometimes there are more basic approaches that solves the equation where using the quadratic root formula would be an overkill.

For example, the SQUARE ROOT PROPERTY: Given

$$x^2 = A$$

with A positive, there are two solutions:

$$x = \pm\sqrt{A}$$

4.3.1 Studied Examples and Problems

1. (Quadratic Root Formula)

   (a) $x^2 + x + 1 = 0$

   We first look at its discriminant: $1^2 - 4 \cdot 1 \cdot 1 = 1 - 4 = -3 < 0$ This means we do NOT have real solutions.

   (b) $x^2 = -x + 1$

   We first move everything to one side and get $x^2 + x - 1 = 0$. This time the discriminant is $1^2 - 4 \cdot 1 \cdot (-1) = 1 + 4 = 5 > 0$. So we have two real solutions given as follows

   $$x = \frac{-1 \pm \sqrt{5}}{2}$$

   Thus the two solutions are $\frac{-1 + \sqrt{5}}{2}$ and $\frac{-1 - \sqrt{5}}{2}$

   (c) **Assignment** Solve: $(x+2)(x+3) = 1$, $(3x-4)(x+2) = (2x-5)(x+5)$ (hint for both: foil then move everything to one side)

2. (Square Root Property)

   (a) $(2x - 3)^2 - 121 = 0$

   One can FOIL and then apply the quadratic root formula but that's a waste of time for this one. Observe that, once you move the -121 to the right, you will get

   $$(2x - 3)^2 = 121$$

   Now using the Square Root Property, we get

   $$2x - 3 = \pm\sqrt{121}$$
   $$2x - 3 = \pm 11$$

   So either $2x - 3 = 11$ or $2x - 3 = -11$. In the first option, we have $x = 7$ and in the latter we have $x = -4$.

   (b) **Assignment** Solve: $(2x - 3)^2 - 18 = 0$, $(2x + 3)^2 + 18 = 0$

3. (Substitution) Some equations are not quadratic but they can be treated the way we treat quadratic equations:

   (a) $x^{10} - 5x^5 + 4 = 0$

   Observe that x^{10} is the square of x^5. Thus if we SUBSTITUTE u for x^5 in the original equation, we get

   $$u^2 - 5u + 4 = 0$$

You can apply the quadratic root formula, but you can easily factor the left hand side. Always try to factor first,instead of applying the quadratic root formula:

$$(u-4)(u-1)=0$$

So by zero factor property, we get

$$u=4 \quad or \quad u=1$$

Wait, we are not done yet! What we found is u, what we want to solve for is x. Using our substitution backwards, we get

$$x^5=4 \quad or \quad x^5=1$$

So there are two solutions; $x=\sqrt[5]{4}$ and $x=\sqrt[5]{1}=1$

(b) $(x^2-4)^2-2(x^2-4)+1=0$

If you FOIL, then you will get a degree four equation and we do not know how to solve them (in this class). Again, we will be smart and do the following substitution: v for x^2-4. We get

$$v^2-2v+1=0$$

We can factor the left hand side actually (if you can factor, don't use the quadratic formula, go with the factorization)

$$(v-1)^2=0$$

So (either by zero-factor property or square root property)

$$v-1=0$$

$$v=1$$

Again; we are not done. Remember that v stands for x^2-4. So we have

$$x^2-4=1$$

$$x^2=5$$

Use square root property now

$$x=\pm\sqrt{5}$$

(c) **Assignment** Solve: $2a^{2/3}-11a^{1/3}+12=0, \quad 2(3k-1)^2+5(3k-1)=-2$

4.4 Equations with Radicals

Take an equation (i will refer to this one as ”original equation”) like

$$z = \sqrt{\frac{5z+3}{2}}$$

To free the z on the right hand side from the radical, we take the square of BOTH sides of the equality. We get the following equation (which i will call ”derived equation”):

$$z^2 = \frac{5z+3}{2}$$

Now, some of you (hopefully) will say ”wait a minute, taking square of both hand sides is not one of the *legal* manipulations we can do to an equation” . Yes, that’s completely true. If you look at the beginning of this section, you will see that we are ONLY allowed to add/subtract from both hand sides or multiply/divide both hand sides...NOT to square both hand sides...why do we do it then? Well, our motivation is the following:

the solutions of the original equation are amongst the solutions of the derived equation

The tricky part is that the derived equation, in general, will have more solutions than the original one. So we have to CHECK every solution of the derived one to see if they are solutions of the original one.

Let’s continue our example. Let’s solve the derived equation:

$$z^2 = \frac{5z+3}{2}$$

multiply both sides by 2:

$$2z^2 = 5z + 3$$

Move everything to one side:

$$2z^2 - 5z - 3 = 0$$

Use quadratic root formula:

$$z = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 2 \cdot (-3)}}{2 \cdot 2} = \frac{+5 \pm \sqrt{49}}{4} = \frac{+5 \pm 7}{4}$$

So there two solution $z = \frac{5+7}{4} = 3$ and $z = \frac{5-7}{4} = -1/2$ of the ”derived” equation. Now let’s CHECK both of them to see if they solve the original equation or not:

$$?? \quad 3 = \sqrt{\frac{5 \cdot 3 + 3}{2}}$$

$$3 = \sqrt{9}$$

true

So $x = 3$ is a solution of the original equation. Now let's check the other one:

$$?? \quad -1/2 = \sqrt{\frac{5\cdot(-1/2)+3}{2}}$$

$$false$$

We don't even need to compute the right hand side to see that it is false. Do you see why? Because the square root of a number cannot be negative, in particular the right hand side cannot be equal to $-1/2$. So $-1/2$ is NOT a solution of the original equation.

The conclusion is that our original equation has only one solution, namely $x = 3$.

Assignment Solve $\sqrt{5-x} - x - 1 = 0$, $\sqrt{k+2} - \sqrt{k-3} = 1$ (hint: apply the method twice)

4.5 Equations with Rational Expressions

When we have equations with polynomial denominators, we call it a rational equation. We have to be careful when we try to get the solutions of such an equation because we need to avoid numbers which make a denominator 0.

4.5.1 Studied Examples and Problems

Let's solve the equation:

$$\frac{2}{3x+1} = \frac{1}{x} - \frac{6x}{3x+1}$$

We FIRST start with identifying what x *cannot* be:

$$3x+1 \neq 0 \quad and \quad x \neq 0$$

Thus

$$x \neq \frac{-1}{3} \quad and \quad x \neq 0$$

Now we start solving for x: I will put everything to common denominator and will combine the two terms n the right hand side:

$$\frac{(2)(x)}{(3x+1)(x)} = \frac{(1)(3x+1)}{(x)(3x+1)} - \frac{(6x)(x)}{(3x+1)(x)}$$

$$\frac{2x}{(3x+1)(x)} = \frac{3x+1-6x^2}{(x)(3x+1)}$$

We have two fractions that are equal to each other with the same denominator: this means their numerators should be the same as well!

$$2x = 3x + 1 - 6x^2$$

Move everything to the left now:

$$6x^2 - 3x - 1 + 2x = 0$$

$$6x^2 - x - 1 = 0$$

Use the quadratic root formula now and you will find that

$$x = \frac{-1}{3} \quad or \quad x = \frac{1}{2}$$

Note that the first solutions is one of those "bad guys" that we should avoid, so I will not (you shouldn't either)count $x = \frac{-1}{3}$ as a solution. Thus we have only one solution, namely $x = \frac{1}{2}$.

Assignment Solve $\frac{5}{x-4} - \frac{3}{x-1} = \frac{x^2-1}{x^2-5x+4}$, $\quad \frac{2}{k^2+k-6} + \frac{1}{k^2-k-2} = \frac{4}{k^2+4k+3}$

5 Answers

1.1	25/24, 14/15, 6/13, 50/42
2.3.1	$b^{-19/12}$, $4m^{7/3} - 16m^{4/3}$
2.3.2	$2\sqrt{10}$, $2\sqrt[3]{5}$, $2\sqrt[4]{6}$, $2\sqrt[5]{3}$, $3\sqrt[3]{3}$, 3
2.3.3	x^3, x^7, $y^5\sqrt[4]{y}$, x^{20} ,$x^{33}\sqrt[3]{x^2}$
2.3.4	$2x^4\sqrt{7x}$, $3y^7x^9\sqrt[3]{2x^2}$, $y^5z^2\sqrt[4]{72yz}$, $2x^{20}t^7\sqrt[5]{4t}$
2.3.5	$(4 + 3xy)\sqrt[3]{xy^2}$, $(2m^2p - 3m^2p)\sqrt[4]{mp^2}$
2.3.6	$\sqrt{6}/3$, $\sqrt[3]{18}/3$, $1/2$, $3 - \sqrt{6}$, $5 - 2\sqrt{6}$
3.2.1	$4x(2x^2 - 1)$, $5xy(3x^2y - 2xy^{3/2} + z)$
3.2.2	$(q^2 + 5)(p^2 - 2)$, $(m + 4)(m^2 - 6)$
3.2.3	$(x - 6)(x + 2)$, $(k - 5)(k - 6)$, $(4p)(p + 8)(p - 2)$
4.1.1	$x = 2$, $t = -17/44$
4.2.1	$\{\ -12, 4\ \}$, $\{\ -5/6, 1, -1\ \}$
4.3.1.1	$\{\ (-5 + \sqrt{5})/2, (-5 - \sqrt{5})/2\ \}$, no real solutions
4.3.1.2	$\{\ (3 + 3\sqrt{2})/2, (3 - 3\sqrt{2})/2\ \}$, no real solutions
4.3.1.3	$\{\ 4^3, (3/2)^3\ \}$, $\{\ 1/6, -1/3\ \}$
4.4	$\{\ 1\ \}$, $\{\ 7\ \}$
4.5.1	$\{\ -2\ \}$, $\{\ 13\ \}$