EXAM 2 REVIEW

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3. Linear Systems and Matrices

3.2. **Matrices and Gaussian Elimination.** At this point in the course, you all have had plenty of practice with Gaussian Elimination. Be able to row reduce *any* matrix you're given. My advice to you concerning not making mistakes is the following:

(1) *Avoid* fractions! Use least common multiples rather than deal with them. Computers don't make mistakes adding fractions, we do.

(2) Take small steps. When I do these problems, you will never see me write down the operation "$\rightarrow^{-2R1+3R2}$". Instead I would break this up into three steps:

$$\begin{array}{ll} 1.) & \rightarrow^{R2=3R2} \\ 2.) & \rightarrow^{R1=-R1} \\ 3.) & \rightarrow^{R2=R1+R2}. \end{array}$$

In fact you can kind of cheat and do both operations 1 and 2 at once. This isn't an elementary row operation, but it doesn't hurt to do that.

(3) Use back substitution after you have your matrix in diagonal form. You don't need your matrix in *reduced* row echelon form to find a solution, just row echelon form.

What I mean by this third piece of advice is the following. Suppose you've already performed the following row operations:

$$\left[\begin{array}{c|c} A & \vec{b} \end{array}\right] \rightarrow^{\text{row ops}} \left[\begin{array}{ccc|c} 4 & 2 & 0 & 0 \\ 0 & 6 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right].$$

This is now in row echelon form, but not reduced row echelon form (see next section) because the off diagonal entries are non zero. But as far as we care, this is enough information to solve for all the variables. Write down the equation described by the 3rd row: $2z = 2$, find $z = 1$. Then write down the equation described by the 2nd row: $6y + z = 1$, and solve for y.

For practice see problems 11, 12 and 23, 24. Practice enough of 11-18 until you don't have to think about doing these problems, it just becomes mechanical.

3.3. **Reduced Row-Echelon Matrices.** The most important theorem from this section is

Theorem 1 (The Three Possibilities Theorem)**.** *Every linear system has either a unique solution, infinitely many solutions or no solutions.*

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Exercise: If A is an $n \times n$ matrix, what are the possible outcomes of solving $A\vec{x} = \vec{0}$? *Hint:* What could A's reduced row echelon form look like?

Know how to put a matrix in reduced row echelon form. Usually if you're interested in solving an equation you don't do this, but this vocabulary term could come up and you should know it. For practice try problems 1-4.

3.4. **Matrix Operations.** Know if and when multiplication/addition is defined for matrices and how to do it. These are easy problems to do, but in case they show up on the exam you want to make sure you don't make any arithmetic mistakes! I'm a bit reluctant to suggest problems from this section, but you should be able to do any of 1-16 with your eyes closed.

See problems 17,18,19. Do these look familiar now? Can you produce a vector/vectors that uniquely describe the kernel of a particular matrix for each of these problems? What are the dimensions of these linear subspaces? (*Hint:* the dimension is the number of free variable that are necessary to describe the space).

3.5. **Inverses of Matrices.** The important definition.

Definition. *An $n \times n$ matrix A (square matrix only!) is said to be* invertible *if there exists a matrix B such that $AB = BA = I_{n\times n}$ where $I_{n\times n}$ is the identity matrix.*

Fact. Inverses, when they exist, are unique. This is a very nice feature to have when you define something! You wouldn't want to get in a fight with your friend over who found the 'better' inverse. We denote *the* (unique) inverse matrix of A by A^{-1}.

You DO NOT need to be prepared to find the inverse of a matrix using row operations! Prof. Bertrand said this type of problem is too long for an exam situation.

One problem you might run into is given two matrices, are they inverses of each other? To check this you need to know how to multiply two matrices together and the definition of the inverse. The following problem is short enough to appear on an exam.

Exercise: If A and B are invertible matrices, is AB an invertible matrix? If so, what's the inverse matrix? Can you prove this?

3.6. **Determinants.** We'll first begin with an extremely important fact. If A is a square matrix, then A is invertible if and only if $\det(A) \neq 0$. You can add this to theorem 7 given on page 193. See your notes from discussion section about what I had to suggest in how to go about finding the determinant of a matrix. You can always do this through cofactor expansion, the thing to keep in mind is the checkerboard pattern that shows up when evaluating the sign of each coefficient.

See problems 1,2. I don't imagine you'll be asked to evaluate the determinant for any large (i.e. larger than 4×4) matrix.

One other trick to keep in mind is what sort of 'row operations' can you do to a matrix when evaluating the determinant. See property 5 listed in the text. This says you're allowed to add a multiple of any row to another row and this doesn't change the determinant. You have to be *very* careful not to misuse this property! This doesn't mean you can arbitrarily multiply a row by a number like we're used to with systems of equations.

See problems 7, 9.

Exercise: If A is an $n \times n$ matrix, express $\det(\lambda A)$ in terms of λ and the matrix A. *Hint:* The answer is not $\lambda \det(A)$.

4. Vector Spaces

4.2. **The Vector Space $\mathbb{R}^n$ and Subspaces.** If we have a vector space V, and we have a subset $W \subset V$, a natural question to ask is whether or not W itself forms a vector space. This means it needs to satisfy *all* the properties of a vector space that are listed on page 236 of your text. The bad news is this is quite a long list, but the good news is we don't have to check every property on the list, because most of them are inherited from the original vector space V. In short, in order to see if W is a vector space, we need only check if W passes the following test.

Theorem 2. *If V is a vector space and $W \subset V$ is a non-empty subset, then W itself is a vector space if and only if it satisfies the following two conditions:*

(1) **Additive Closure** *If $\vec{a} \in W$ and $\vec{b} \in W$, then $\vec{a} + \vec{b} \in W$.*
(2) **Multiplicative Closure** *If $\lambda \in \mathbb{R}$ and $\vec{a} \in W$, then $\lambda\vec{a} \in W$.*

The statement of this theorem has the term 'non-empty' as one hypothesis for the theorem to be true. In *most* applications of this theorem, we actually replace the statement non-empty with requiring that $\vec{0} \in W$. I.e., the theorem from above is equivalent to the following theorem.

Theorem 3. *If V is a vector space and $W \subset V$, then W itself is a vector space if and only if it satisfies the following two conditions:*

(1) **Additive Closure** *If $\vec{a} \in W$ and $\vec{b} \in W$, then $\vec{a} + \vec{b} \in W$.*
(2) **Multiplicative Closure** *If $\lambda \in \mathbb{R}$ and $\vec{a} \in W$, then $\lambda\vec{a} \in W$.*
(3) **Non Empty** $\vec{0} \in W$.

Note that these are two properties that are on the long laundry list of properties we require for a set to be a vector space.

Example Consider $W := \{\vec{a} = (x, y) \in \mathbb{R}^2 : \ x = 2y\}$. Since $W \subset \mathbb{R}^2$, we may be interested if W itself forms a vector space. To answer this question we need only check two items:

(1) Additive Closure: An arbitrary element in W can be described by $(2y, y)$ where $y \in \mathbb{R}$. Let $(2y, y), (2z, z) \in W$. Then $(2y, y) + (2z, z) = (2y + 2z, y + z)) \in W$ since $2y + 2z = 2(y + z)$.
(2) Multiplicative Closure: We need to check if $\lambda \in \mathbb{R}$, and $\vec{a} \in W$, then $\lambda\vec{a} \in W$. Again, an arbitrary element in W can be described by $(2y, y)$ where $y \in \mathbb{R}$. Let $\lambda \in \mathbb{R}$ and $(2y, y) \in W$. Then $\lambda(2y, y) = (2\lambda y, \lambda y) \in W$ since the first coordinate is exactly twice the second element.

Note: it is possible to write this set as the kernel of a matrix. In fact, you can check that $W = \ker(A)$, where $A_{1\times 2} = \begin{bmatrix} -2 & 1 \end{bmatrix}$. We actually have a theorem that says the kernel of any matrix is indeed a linear subspace.

Example Consider $W := \{\vec{a} \in \mathbb{R}^3 : \ z \geq 0\}$. In order for this to be a linear subspace of $\mathbb{R}^3$, it needs to pass two tests. In fact, this set passes the additive closure test, but it **doesn't** pass multiplicative closure! For example, $(0, 0, 5) \in W$, but $(-1) \cdot (0, 0, 5) = (0, 0, -5) \notin W$.

Definition. *If $A_{m\times n}$ is a matrix, we define* $\ker(A) := \{x \in \mathbb{R}^n : \; Ax = 0\}$*. This is also called the* nullspace *of A.*

Note that $\ker(A)$ lives in $\mathbb{R}^n$.

Definition. *If $A_{m\times n}$ is a matrix, we define $Image(A) := \{y \in \mathbb{R}^m : \; Ax = y \text{ for some } x \in \mathbb{R}^n\}$. This is also called the* range *of A.*

Note that Image(A) lives in $\mathbb{R}^m$.

Theorem 4. *If $A_{m\times n}$ is a matrix, then* $\ker(A)$ *is a linear subspace of $\mathbb{R}^n$ and Image(A) is a linear subspace of $\mathbb{R}^m$.*

Exercise: show this theorem is true. In lieu of section 4-4, I think likely candidates from this section will be proving a set of element is not a vector space. For example see problems 7, 9, 10, 13.

Problems 15-22 serve as excellent practice for leading up to section 4-4, but you should get the gist after doing problems from that section.

4.3. **Linear Combinations and Independence of Vectors.** If we have a collection of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, we can form many vectors by taking linear combinations of these vectors. We call this space the *span* of a collection of vectors, and we have the following theorem:

Theorem 5. *If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a collection of vectors in some vector space V, then*

$$span\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} := \{\vec{w} : \; \vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k, \text{ for some scalars } c_i \in \mathbb{R}\}$$

is a linear subspace of V.

For a concrete example, we can take two vectors $\vec{v}_1 = (1, 1, 0)$ and $\vec{v}_2 = (1, 0, 0)$ which both lie in $\mathbb{R}^3$. Then the set $W = \text{span}\{(1, 1, 0), (1, 0, 0)\}$ describes a plane that lives in $\mathbb{R}^3$. This set is a linear subspace by this previous theorem. In fact, we can be a bit more descriptive and write $W = \{(x, y, z) \in \mathbb{R}^3 : \; z = 0\}$.

If we continue with this example, it is possible to write W in many other ways. In fact, we could have written

$$W = \text{span}\{(1, 1, 0), (1, 0, 0), (-5, 1, 0)\} = \text{span}\{(-10, 1, 0), (2, 1, 0)\}.$$

These examples illustrate the fact that our choice of vectors need not be unique. What is unique, is the *least number* of vectors that are required to describe the set. In fact this is so important we give it a name, and call it the *dimension* of a vector space. This is the content of section 4.4. In our example, $\dim(W) = 2$, but right now we don't have enough tools to show this.

In order to make this statement 'least', precise, we need to introduce following important definition.

Definition Vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ are said to be *linearly independent* if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = 0$$

for some scalars c_i, it must follow that $c_i = 0$ for each i.

OK, so definitions are all fine and good, but how do we check if vectors are linearly independent? The nice thing about this definition is it always boils down to solving a linear system.

Example As a concrete example, let's check if vectors $\{\vec{v}_1, \vec{v}_2\}$ linearly independent where $\vec{v}_1 = (4, -2, 6, -4)$ and $\vec{v}_2 = (2, 6, -1, 4)$.

We need to solve the problem

$$c_1\vec{v}_1 + c_2\vec{v}_2 = 0.$$

This reduces to asking what are the solutions to

$$c_1 \begin{bmatrix} 4 \\ -2 \\ 6 \\ -4 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 6 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can write this problem as a matrix equation $A\vec{c} = \vec{0}$ where

$$A = \begin{bmatrix} 4 & 2 \\ -2 & 6 \\ 6 & -1 \\ -4 & 4 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

and solve this using Gaussian elimination.

$$\left[\begin{array}{cc|c} 4 & 2 & 0 \\ -2 & 6 & 0 \\ 6 & -1 & 0 \\ -4 & 4 & 0 \end{array}\right] \rightarrow^{\text{row ops}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus $c_1 = c_2 = 0$ is the *only* solution to this problem, and so these two vectors are linearly independent.

To demonstrate that a collection of vectors are not linearly independant, it suffices to find a non-trivial combination of these vectors and show they sum to $\vec{0}$. For example, see example 6 in the textbook.

We have another method for checking if vectors are linearly independent. This method is more complicated to apply so I encourage you become familiar with using Gaussian Elimination (row operations) for checking independence. But to be complete, I'll include this other method as well. Essentially what happened in this last problem was we were able to do row operations to a matrix, and get the 'identity' matrix in a part of it. Being row equivalent to the identity matrix is equivalent to being invertible and this is equivalent to having a non-zero determinant. What makes this tricky is *what* part of what matrix are we considering because determinants are only defined for square matrices. We'll do the special case first:

Theorem 6. *Independance of n Vectors in $\mathbb{R}^n$ The n vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ are linearly independent if and only if* $\det(A) \neq 0$ *where (as usual)*

$$A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \\ | & | & & | \end{bmatrix},$$

A is the (square $n \times n$) matrix given by putting the vectors $\vec{v}_i$ as collumns.

Note: this theorem currently only applies to a collection of n vectors in $\mathbb{R}^n$, but we can easily extend this to other collections of vectors. We'll do this in a minute. What this theorem doesn't give us is what combination gives us a sum that's $\vec{0}$.

I do think seeing a proof of this is instructive. It tells you why we care about even taking determinants here, as well being a good illustration for how one goes about checking for independence.

Proof: (If $\det(A) \neq 0$*, then the vectors are independent).* Suppose $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$ for some scalars c_i. Then as usual, this leads to solving the system $A\vec{c} = \vec{0}$ where

$$A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix}, \qquad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Since $\det(A) \neq 0$, we know A is invertible which also means A is row equivalent to the identity matrix. This means we can do row operations to this and turn the system into:

$$[A \mid 0\,] = \left[\begin{array}{cccc|c} | & | & & | & 0 \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n & 0 \\ | & | & & | & 0 \end{array}\right] \rightarrow^{\text{row ops}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & 0 \\ 0 & \cdots & 1 & 0 & \\ 0 & \cdots & 0 & 1 & 0 \end{array}\right]$$

Evidently, $c_1 = c_2 = \cdots = c_n = 0$, so the vectors are linearly independent. □

To summarize. Know the definition of linear independence. Know that checking for independence always results in asking what are the solutions to the equation $A\vec{c} = \vec{0}$. If $\vec{c} = \vec{0}$ is the *only* solution, then they are independent, and if there's a non-zero $\vec{c}$ that solves this, they are dependent. Finding $\vec{c}$ gives you the coefficients $c_1, c_2, \dots$ that demonstrate linear dependence. Gaussian elimination (row operations) is a method that will give you the coefficients.

Exercise Is the set of vectors $\{(0,0,1),(1,0,1),(0,0,0)\}$ linearly independent?

Exercise Is the set of vectors $\{(1,1,1),(1,5,-1),(-10,17,0),(0,1,0)\}$ linearly independent? *Hint:* There is a one line solution to this, or you can write down the full linear system and solve it.

Exercise

5. Bases and Dimension for Vector Spaces

We'll begin with the major definition of the section.

Definition. *A collection of vectors* $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ *are a* basis *for a vector space* V *if they satisfy*

(1) $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ *are linearly independent.*
(2) $V = span\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$.

Fact: The number of vectors in any basis for a finite dimensional vector space is unique. We define $\dim(V) = n$ where n is the number of vectors in a basis.

Theorem 7. *If* $\dim(V) = n$*, and* $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ *is a collection of* n *linearly independent vectors, then* S *is a basis for* V*.*

Fact: $\dim(\mathbb{R}^n) = n$. This is a deep, non-trivial result that depends on the previous theorem! Why is this true? For $\mathbb{R}^3$, we have the basis $S = \{(1,1,1),(1,1,0),(1,0,0)\}$ and there are 3 vectors here.

I'm guessing a problem very similar to your homework problems will show up with a high probability. All of your problems asked you to find a basis for the kernel of a matrix. Know how to do this.

Practice 12-20 until you know how to do this!

5.1. **Good Luck!**