**Chapter 7 Notes**
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## 6. Eigenvalues and Eigenvectors

A scalar[1] $\lambda \in \mathbb{C}$ is an *eigenvalue* of the $n \times n$ matrix $A$ if there exists a non-zero vector $\vec{v} \in \mathbb{R}^n$ such that

$$A\vec{v} = \lambda\vec{v}.$$

If we subtract $\lambda\vec{v}$ from both sides, the above equation is equivalent to

$$(A - \lambda I)\vec{v} = \vec{0}$$

for some non-zero $\vec{v}$. Since $\vec{v} \neq \vec{0}$, we require $A - \lambda I$ to be non-invertible, otherwise the only solution to the last equation is $\vec{v} = \vec{0}$. That is we can perform the following algorithm to find every eigenvalue and eigenvector:

(1) Set $\det(A - \lambda I) = |A - \lambda I| = 0$ and solve for $\lambda$. This gives us every possible eigenvalue.
(2) For each eigenvalue computed in step (1), solve $(A - \lambda I)\vec{v} = \vec{0}$ for $\vec{v}$. Note that this is just a restatement of the equation $A\vec{v} = \lambda\vec{v}$.

## 7. Linear Systems of Differential Equations

We oftentimes run into matrices that do not have 'real'-valued eigen-values. In such cases, we have two options: 1) give up and say this matrix has no eigenvalues, or 2) use complex numbers to factor the characteristic polynomial. The term imaginary is a bit of a misnomer because there is nothing 'imaginary' about complex (imaginary) eigenvalues, so I will try to stick to the term complex.

**7.3.** In this section, we are interested in solving the problem

$$A\vec{x} = \dot{\vec{x}} \tag{1}$$

where $A$ is an $n \times n$ square matrix. (I'm going to use the notation $\frac{d}{dt}\vec{x} \equiv \dot{\vec{x}}$ because it's easier to type. This is very common notation, but not used in our textbook).

7.3.1. *Preliminary Theory.* If $\{\vec{x}_1, \ldots, \vec{x}_k\}$ are solutions to $A\vec{x} = \dot{\vec{x}}$, then so is

$$\vec{x}(t) = c_1\vec{x}_1(t) + \cdots + c_k\vec{x}_k(t). \tag{2}$$

Our goal is to find as many solutions $\vec{x}_i(t)$ as possible, then take linear combinations to form the general solution.

The function $\vec{x}_i(t) = e^{\lambda_i t}\vec{v}_i$ solves $A\vec{x} = \dot{\vec{x}}$ where $\lambda_i$ is any eigenvalue of $A$ with eigenvector $\vec{v}_i$. You can see this if you just plug it into the equation:

$$A\vec{x}_i(t) = e^{\lambda_i t}A\vec{v}_i = e^{\lambda_i t}\lambda_i\vec{v}_i = \frac{d}{dt}(e^{\lambda_i t}\vec{v}_i).$$

Taking linear combinations of the solutions $\vec{x}_i$, equation (**??**) becomes

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_k e^{\lambda_k t}\vec{v}_k. \tag{3}$$

If we have $k = n$ distinct eigenvalues, then this completely solves the problem. In section 7.5 we handle the case when we don't have 'enough' eigenvectors from these eigenvalues.

[1]Sometimes we take scalars from $\mathbb{R}$, and sometimes we take them from $\mathbb{C}$ - the definitions and theory we learn is identical for either choice.

Note: that this theory doesn't care if our eigenvalues are real or complex - the above formula holds for *any* eigenvalue/eigenvector pair!

7.3.2. *Complex Valued Solutions.* Suppose we have a complex-valued solution $\vec{y}(t) = \vec{x}_1(t) + i\vec{x}_2(t)$. If we plus this into the differential equation, we can see that the real and imaginary parts ($Re(\vec{y}) = \vec{x}_1$ and $Im(\vec{y}) = \vec{x}_2$) both solve the differential equation. On one hand,

$$A\vec{y} = A(\vec{x}_1(t) + i\vec{x}_2(t)) = A\vec{x}_1(t) + iA\vec{x}_2(t).$$

On the other hand,

$$\dot{\vec{y}} = \dot{\vec{x}}_1 + i\dot{\vec{x}}_2.$$

Since $A\vec{y} = \dot{\vec{y}}$, the real and imaginary parts [2] must be equal, and hence

$$\dot{\vec{x}}_1 = A\vec{x}_1, \qquad \dot{\vec{x}}_2 = A\vec{x}_2$$

are two solutions of (1). So from one complex-valued solution the problem, we obtain two real-valued solutions.

If we have a complex eigenvector $\vec{v}$ with associated eigenvalue $\lambda = \alpha + i\beta$, then we know from equation (2) that the function

$$\vec{y}(t) = e^{\lambda t}\vec{v} = e^{\alpha t}e^{i\beta t}\vec{v} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))\vec{v}$$

is a solution to the problem. Hence we have two solutions (they will be linearly independent) from one complex eigenvalue/eigenvector pair by setting

$$\vec{x}_1(t) := Re(\vec{y}) \quad \text{and} \quad \vec{x}_2(t) := Im(\vec{y}).$$

### 7.4. Second-Order Systems and Mechanical Vibrations.

**7.5. Multiple Eigenvalue Solutions.** Suppose we have a $3 \times 3$ matrix $A$ with eigenvalues $\lambda = 2$ (mult. $\times 2$) and $\lambda = 1$ (mult. $\times 1$). With the eigenvalue $\lambda = 1$, we expect exactly 1 eigenvector $\vec{v}_1$. With the eigenvalue $\lambda = 2$, we can have either one or two eigenvectors.

If $\lambda = 2$ produces two eigenvectors $\vec{v}_2, \vec{v}_3$, then we say this eigenvalue is *complete* and the solution to (1) is given by equation (3):

$$\vec{x}(t) = c_1 e^t \vec{v}_1 + c_2 e^{2t}\vec{v}_2 + c_3 e^{2t}\vec{v}_3.$$

This is the good case: $A$ has a complete set of eigenvectors and hence $A$ is also diagonalizable.

If $\lambda = 2$ produces only one eigenvector (it's going to produce at least one!), then we say this eigenvalue is defective. The number of 'missing' eigenvalues is its degeneracy $d = 1$. The way to handle this case is to perform the following algorithm:

(1) Solve $(A - 2I)^2\vec{u}_2 = \vec{0}$ for $\vec{u}_2$. If this is the zero matrix, then *any* vector $\vec{u}_2$ works, and so you can usually get away with choosing the easiest one: $\vec{u}_2 = (1, 0, 0)$.

(2) Set $\vec{u}_1 = (A - 2I)\vec{u}_2$. Doing this actually forces $\vec{u}_1$ to be an eigenvector since then
$$(A - 2I)\vec{u}_1 = (A - 2I)^2\vec{u}_2 = \vec{0}.$$
and hence $A\vec{u}_1 = \lambda\vec{u}_1$.

(3) Consider the two solutions $\vec{x}_1(t) = e^{2t}\vec{u}_1$ and $\vec{x}_2(t) = e^{2t}(t\vec{u}_1 + \vec{u}_2)$. The general solution is given by equation (2) with $\vec{x}_3 = e^t\vec{v}_1$.

[2]Complex numbers, just like vectors are equal if and only if each component is equal.