CHAPTER 1

First-Order Differential Equations

1. Diff Eqns and Math Models

Know what it means for a function to be a ‘solution’ to a differential equation. In order to figure out if $y = y(x)$ is a ‘solution’ to the differential equation, we plug this into the differential equation and see if it solves it.

For example, in algebra we may be faced with an equation like

$$x^2 - 5x + 10 = 2x.$$

We can verify that $x = 2$ is a solution to this equation by plugging it in and verifying that it solves the equation. Here is my proof that $x = 2$ is the solution to the previous equation:

$$\begin{aligned} LHS &= x^2 - 5x + 10 = 2^2 - 5(2) + 10 = 4 - 10 + 10 = 4. \\ RHS &= 2x = 2(2) = 4. \end{aligned}$$

Since $LHS = RHS$, $x = 2$ is a solution. When verifying your solution, do NOT manipulate both sides of the equation. For example, the following is NOT a valid proof that $x = 2$ is a solution:

$$\begin{aligned} 2^2 - 5(2) + 10 &= 2(2) \\ 4 - 10 + 10 &= 4 \\ 4 - 10 &= 4 - 10 \\ -6 &= -6. \end{aligned}$$

For practice reviewing derivatives, you can look at $1, 4, 7, 10$ and $17, 20, 23$.

2. Integrals as General and Particular Solutions

This section is intended to be more practice with integration. You should be able to solve 1–8 and 10 without thinking about them too much. Remember integration by parts is your friend:

$$\int u\ dv = uv - \int v\ du.$$

Try using this tool on problem 10.

Another integration technique that is extremely useful is Partial Fractions. In order to do the partial fractions setup for the function $f(x) = \frac{1}{x^3+3x^2}$ we first need to completely factor the denominator:

$$f(x) = \frac{1}{x^3 + 3x^2} = \frac{1}{x^2(x+3)}.$$

Once you have all the factors, just set up the partial fractions. If any factor has degree higher than 1, you need to put a polynomial that has one less degree on the numerator:

$$\frac{1}{x^2(x+3)} = \frac{A}{x+3} + \frac{Bx+C}{x^2}.$$

The velocity problems are covered in much more detail in section 2-3. Problems 11–18 are good practice with integration and require you to know the relations between a, v and x. You should be able to do these without thinking about the problems too much.

3. Slope Fields and Solution Curves

You have one theorem whose statement you need to memorize:

THEOREM 1. *If $f(x, y)$ and the partial derivative $\frac{\partial f}{\partial y}(x, y)$ is continuous 'near' the point (x_0, y_0), then the initial value problem*

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

has a unique solution on some (possibly small) interval containing x_0.

If the hypotheses of the theorem are not satisfied, then anything goes. You may have one solution, infinitely many solutions or no solutions whatsoever.

For practice with this theorem, you may want to consider trying problems: $11, 12, 13, 14, 17, 18$.

As a variation on this theorem, you can drop the hypothesis on the continuity of $\frac{\partial f}{\partial x}$ and obtain existence (possibly without uniqueness). See http://en.wikipedia.org/wiki/Peano_existence_theorem.

4. Separable Equations and Applications

The technique of separating variable allows us to solve a whole new class of problems that aren't covered in the standard 222 course. Any problem from 1–28 will be extremely good practice for the exam.

In addition, this section introduces a few new models not previously covered:

(1) Population Growth: $\frac{dP}{dt} = kP$;. $k > 0$ is the growth constant.
(2) Radioactive Decay: $\frac{dN}{dt} = -kN$; $k > 0$ is the decay constant.
(3) Newton's Law of Cooling: $\frac{dT}{dt} = k(A - T)$; $k > 0$ is a constant that has to do with how well insulated the system is. $A \equiv \textit{constant}$ is the ambient air temperature.

Now that we know about phase diagrams, you should be able to sketch one of these for each of these equations. Note that the first two equations are essentially the same, $\frac{dy}{dt} = \textit{const} \cdot y$, after you're supplied with enough initial conditions, you'll be able to determine the correct sign for the constant.

For practice, you may want to consider trying problems 33–36, 43, 49, 65.

5. Linear First-Order Equations

In this section we learned how to solve any linear first order differential equation. These equations are anything that can be written in what I call so called 'standard form':

$$\frac{dy}{dx} + P(x)y = Q(x).$$

In order to solve these equations, we use the INTEGRATING FACTOR METHODS. Memorize the method given on page 47. You should probably add in a step 0, which says 'write in standard form'. The most important thing to remember is how to compute the integrating factor: $\rho = \exp(\int P(x)\ dx)$.

Practice problems 1–19 (every third) until you become comfortable with this method. It's VERY important to understand this method because it is one of the only two methods that we learn in the first two chapters.

The integrating factor method allows us to solve Mixing Problems. The standard setup is given by the picture on page 51. I think it helps to do the dimensional analysis to come up with the terms:

$$\frac{dx}{dt} = \text{stuff in} - \text{stuff out}.$$

Since $\left[\frac{dx}{dt}\right] = \frac{\text{g salt}}{\text{sec}}$, we know that 'stuff in' has units of $\frac{\text{g salt}}{\text{sec}}$. Therefore if we take $[r_i] = \frac{\text{g salt}}{\text{L}}$ and multiply it by $[c_i] = \frac{\text{L}}{\text{sec}}$ we'll have the correct units:

$$\frac{dx}{dt} = r_i c_i - r_o c_o.$$

The rate in, r_i and c_i is usually given to us. To find c_o we need to compute this using what we know about the problem. The volume $V(t)$ of the tank can usually be explicitely computed, and hence $c_o = \frac{x}{V}$. The differential equation becomes:

$$\frac{dx}{dt} = r_i c_i - r_o c_o = r_i c_i - \frac{x(t)}{V(t)} r_o.$$

For practice with this problem redo your homework problem 33, and try solving 36 and 37 on your own.

CHAPTER 2

Mathematical Models and Numerical Methods

1. Population Models

Here we encounter a more sophisticated population equation that's called the logistic's equation. It can be found on page 79, equation 3:

$$\frac{dP}{dt} = kP(M - P).$$

One can think of this as an extension to the unbounded population growth model: $\frac{dP}{dt} = kP$ by subtracting another term that is proportional to P^2. Thus, when P is small, the kP term dominates, and when P is large, the P^2 term dominates.

From a phase diagram, you should be able to immediately see what the limiting population is.

For review of partial fractions, you may want to consider looking at problems 1–8. For practice with some population models, try problems 9 and 21.

2. Equilibrium Solutions and Stability

When we're studying first order differential equations of the form

$$\frac{dx}{dt} = f(x)$$

where $f = f(x)$ is a function of x only, (note that t is the **independent** variable, and in this context, x is the **dependent** variable) we can oftentimes derive qualitative information about what happens as the solution evolves over time from a given initial condition x_0.

Know how to find a **critical point**. (set $f(x) = 0$, and solve for x). Know how to determine if your critical point is stable/unstable/semistable.

For practice, try problems 1, 3, 5 and 9. Know how to analyze the stability and long term behavior for the logistics equation from section 2-1 as well as the variation that includes the harvesting parameter $h \geq 0$:

$$\frac{dP}{dt} = kP(M - P) - h.$$

What qualitative behavior changes as h is increased? Is there a special point where h drastically changes the behavior of the solutions?

3. Acceleration-Velocity Models

3.1. Gravity. Newton's second law (abbreviated N2L) states that $F = ma$ where F is the force applied on the object, m is the mass of the object, and $a = \frac{dv}{dt} = \frac{d^2y}{dt^2}$ is the objects acceleration. Recall the relations between position, velocity and acceleration.

In general F is actually a vector, so it has velocity and magnitude. Since we have only ever worked in 1-dimension, the only options for F are positive or negative.

When N2L is applied to gravity, we have $|F_G| = ma$. One of the fundamental assumptions concerning gravity is $|F_G| = \frac{GM \cdot m}{d^2}$ where d is the distance between the two objects, G is the gravitational constant and M, m are the masses of the two bodies. After dividing by m, we have

$$|a(t)| = \frac{GM}{d^2},$$

so that in fact the acceleration due to the force of gravity is independent of the objects mass!

When $d \approx R$ doesn't vary too much (i.e. for an object which stays near the surface of the earth), then we can say that $\frac{GM}{d^2} \approx g$ is a constant. Hence:

$$\frac{dv}{dt} = -g.$$

The minus sign is to account for the direction of the force of gravity. When we're using units of ft, we have that $g = 32$. When we're measuring distance in terms of meters, $g = 9.8$. For practice, I suggest you review your homework problems: $25, 30$. In addition I suggest looking at problems 3 and 20 for extra practice.

4. Numerical Approximation: Euler's Method

Euler's method is the most basic and fundamental Numerical Technique for solving differential equations. I should emphasize that when one studies any real world applied problem, one usually needs to resort to numerical techniques for solving a differential equation.

The best way to derive Euler's method is to start with what we'll call the **discrete derivative**. To see where this comes from recall the definition of the derivative:

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}.$$

From here it makes sense that $y'(x) \approx \frac{y(x+h)=y(x)}{h}$ when h is very small. Now if we write $y_{n+1} = y(x_n + h)$; $y_n = y(x_n)$; $x_{n+1} = x_n + h$, we have the approximation

$$y'(x_n) \approx \frac{y_{n+1} - y_n}{h}.$$

If we're trying to solve the equation

$$\frac{dy}{dx} = f(x, y)$$

then we just set this **discrete derivative** equal to the right hand side function f:

$$\frac{y_{n+1} - y_n}{h} = f(x_n, y_n).$$

Solving this equation for y_{n+1} gives us the updating formula. Initialize y_0 and x_0 from the initial conditions. For $n = 0, 1, 2, 3, ...$ do:

$$y_{n+1} = y_n + hf(x_n, y_n), \quad x_{n+1} = x_n + h.$$

For practice try problems 3 and 6.