WEEK 2 NOTES

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1. A Mixing Problem Example

The standard mixing problem is the following. We wish to measure the amount of 'stuff' (salt) in a well mixed container (pond). We know what's going into the pond, how much salt was initially in the pond, and how fast this stuff is coming out.

The unknown we'd like to solve for is

$$x(t) \equiv \text{amount of salt in tank (grams)},$$

and we're given the following quantities

$$\begin{aligned} r_i, c_i &\equiv \text{rate and concentration in}, \\ r_o &\equiv \text{rate out}. \end{aligned}$$

Let's introduce

$$V(t) \equiv \text{Volume of the tank},$$

and hence we can write the concentration out as $c_o \equiv \frac{x}{V}$.

I think it helps to do the dimensional analysis to come up with a differential equation. We know enough about the time rate of change of the salt to write an equation of the form

$$\frac{dx}{dt} = \text{stuff in} - \text{stuff out}.$$

Since $\left[\frac{dx}{dt}\right] = \frac{\text{g salt}}{\text{sec}}$, we know that 'stuff in' has units of $\frac{\text{g salt}}{\text{sec}}$. Therefore if we take $[r_i] = \frac{\text{g salt}}{\text{L}}$ and multiply it by $[c_i] = \frac{\text{L}}{\text{sec}}$ we'll have the correct units:

$$\frac{dx}{dt} = r_i c_i - r_o c_o.$$

The rate in, r_i and c_i are given to us. To find c_o we need to compute this using what we know about the problem. Since we know what c_o is in terms of the amount of salt, we can write the differential equation as

$$\frac{dx}{dt} = r_i c_i - r_o c_o = r_i c_i - \frac{x(t)}{V(t)} r_o.$$

For a concrete example, let's assume $r_i = \frac{5 \text{ gal}}{\text{min}}, c_i = \frac{2 \text{ g salt}}{\text{min}}$ and $r_o = \frac{6 \text{ gal}}{\text{min}}$. Unlike in your hw problem, the net flow through this tank isn't 0, so this tank will eventually empty. Let's assume the initial volume of the tank is 100 gallons, and there isn't any salt in the tank at the beginning. Hence we can write down how much water is in the tank by

$$V(t) = 100 + t(r_i - r_o) = 100 - t.$$

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The differential equation then becomes

$$\frac{dx}{dt} = 10 - \frac{6}{100-t}x.$$

Is this a linear differential equation?[1] Is it a first order differential equation?[2] How can we solve first order linear differential equations?[3]

2. Integrating Factor Method

The integrating factor method can be used to solve ANY first order linear differential equation. First order means the highest derivative that appears is only a single prime. Linear means the y or y' isn't buried inside another function and they y and y' can't be multiplying each other. I should caution you that although this tool works on any problem of this class, it's not always the most efficient. For example, if you'd like to solve

$$\frac{y'}{t} = sin(t),$$

you might not want to use the integrating factor method.

2.1. **What is the Integrating Factor Method?**

Here is a cookbook formulaic way of applying the integrating factor method:

(0) Write in "standard form." This means make sure the coefficient of y' is 1.
(1) Compute the integrating factor, μ.
(2) Multiply both sides of the equation by μ.
(3) Recognize the new left hand side of the DE as the derivative of a product.
(4) Integrate both sides.

To illustrate this, I'll give you an example. This is problem 18 from section 2-1.

$$ty' + 2y = \sin(t), \quad y(\pi/2) = 1, \quad t > 0.$$

First, write in "standard form":

$$y' + \frac{2}{t}y = \frac{\sin(t)}{t}.$$

Next, we need to compute the 'integrating factor':

$$\mu := \exp\left(\int \frac{2}{t}\, dt\right) = \exp\left(2\ln(t)\right) = t^2.$$

Note that we do not care about *this* constant of integration, so we are free to choose it to be 0. Next, we multiply both sides of the equation by μ:

$$t^2y' + 2ty = t\sin(t).$$

So far note that we haven't done anything! But here's where the cool stuff happens. Assuming we did everything, the left hand side is now the derivative of a product:

$$\frac{d}{dt}\left(t^2y\right) = t\sin(t).$$

Integrating both sides, and using integration by parts for the right hand side ($\int u\ dv = uv - \int v dt$) with $u = t$ and $dv = \sin(t)$ gives

$$t^2y = -t\cos(t) + \sin(t) + C.$$

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[1] yes!
[2] yes!
[3] I don't know! I havn't read section 2.1 yet!

Diving by t^2 gives the general solution:

$$y = -\frac{\cos(t)}{t} + \frac{\sin(t)}{t^2} + \frac{C}{t^2}.$$

Using the initial condition, one can solve for C.