Math 222 – Review problems.

1. Determine

$$\lim_{x\to 0} \frac{\sin x}{e^x - 1}$$

by using the Taylor expansions of e^x, e^{-x} and $\sin x$. Same for

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{\sin^2 x}.$$

2. (i) Show that

$$0 < e^x - 1 - x < \frac{1}{100} \text{ for } 0 < x < 1/10 \ .$$

(ii) Show that for all $x > 0$

$$0 < \int_0^x \arctan(t)\, dt < \frac{x^2}{2}.$$

3. *Study all aspects of this problem well.* Review Taylor's formula with integral remainder $f(x) = T_n^a(x) + R_n^a(x)$ where

$$R_n^a(x) = \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t) dt.$$

Here we assume that the $n+1$st derivative is a continuous function.

(i) Use a substitution in the above integral to prove that the remainder term can also be written as

$$R_n^a(x) = \frac{(x-a)^{n+1}}{n!} \int_0^1 (1-s)^n f^{(n+1)}(a + s(x-a)) ds.$$

(ii) Let m_1 and m_2 be constants so that $m_1 \le f^{(n+1)}(t) \le m_2$ for all t between a and x. Show that

$$\frac{m_1}{n+1} \le \int_0^1 (1-s)^n f^{(n+1)}(a + s(x-a)) ds \ \le \frac{m_2}{n+1}.$$

(iii) If $|f^{(n+1)}(t)| \le M$ for all t between a and x then

$$|R_n^a(x)| \le \frac{M}{(n+1)!} |x-a|^{n+1}.$$

Show that this follows from part (ii).

4. Prove that the Taylor series of $\sin(2x)$ converges to $\sin(2x)$ for all x.

5. (i) Find the Taylor series for $\frac{x}{x+4}$ (in terms of powers of x). Explain why it converges for $|x| < 4$.

(ii) Find the Taylor series of $\frac{1}{(x+4)(x-2)}$ (in terms of powers of x). Explain why it converges for $|x| < 2$.

6. Suppose $f(x) = o(x^2)$ as $x \to 0$. Why is it true that $f(x) = o(x)$ as $x \to 0$?

7. Each of the following expressions $f(x)$ satisfies $\lim_{x\to 0} f(x) = 0$. Find the *largest* non-negative integer n so that the expression is $o(x^n)$ as $x \to 0$.

(i)
$$\cos x - \cosh x.$$

(ii)
$$\cos(x^2) + \cosh(x^2) - 2.$$

(iii)
$$\int_0^x \frac{\sin t}{t} dt.$$

(iv)
$$\int_0^x \frac{\cos(3t^2) - 1}{t^3} dt.$$

(v)
$$\frac{1}{1-x} - 1.$$

Also, for each expression $f(x)$ above find a number m for which the limit $\lim_{x\to 0} \frac{f(x)}{x^m}$ exists (as a number) and is not equal to 0. Then determine this limit.

8. Determine the Taylor polynomial of degree 11 for the following functions (i) $\sin(x^2)$, (ii) $1 + x^2 + e^{3x^2}$, (iii) $\int_0^x (1 + t^2 + e^{3t^2}) dt$

Which theorem do you use that supports your calculation?

9. Approximate e^x by its third order Taylor polynomial (in powers of x) to find an approximate value of
$$\int_0^1 \frac{e^x - 1}{x} \, dx\,.$$
Estimate the error.

10. We expand $2x\cos(x^2)$ in its Taylor series in powers of x. It converges for all x (why?). Write $3x\cos(x^2) = \sum_{n=0}^{\infty} a_n x^n$ and give a formula for a_n. You may have to distinguish several cases.

11. (i) Compute the integrals
$$\int_0^x \frac{3}{\sqrt{5+t^2}} \, dt$$
and
$$\int_0^x t\cosh(t) \, dt\,.$$
(ii) Approximate for small x both integrals by a cubic polynomial and estimate the error (depending on x).

12. Write the following complex numbers in the form $a + bi$ where a and b are real.
(i) $(3+i)(4-2i)$, (ii) $\frac{2+i}{2-i}$, (iii) $\frac{3+i}{2+5i}$, (iv) $(\cos\alpha + i\sin\alpha)^3$, (v) $(1+i)^5$.