

43. Use the substitution $w = 1 + u^2$ with $dw = 2u du$ and then

$$\int_1^2 \frac{u du}{1 + u^2} = \int_{1^2}^{2^2} \frac{1}{2w} dw = \frac{1}{2}(\ln 4 - \ln 1) = \ln 2.$$

44. Same answer as in 43.

45. Use substitution $u = \sin(\theta)$ with $du = \cos \theta d\theta$. Then

$$\int_{\pi/4}^{\pi/3} \sin^2 \theta \cos \theta d\theta = \int_{\sqrt{2}/2}^{\sqrt{3}/2} u^2 du = \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^3 - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3.$$

46. Substitute $u = \ln r$, $du = r^{-1} dr$ and get

$$\int_2^3 \frac{1}{r \ln r} dr = \ln \ln 3 - \ln \ln 2.$$

47. $\int \frac{\sin 2x}{1 + \cos^2 x} dx.$

Substitute $u = 1 + \cos^2 x$, with $du = 2 \cos x(-\sin x)dx$ hence $u = 1 + \cos^2 x$ and the result is $-\ln(1 + \cos^2 x)$.

Thus the function represented by the integral is $-\int u^{-1} du$ with $u = 1 + \cos^2 x$ and the result is $-\ln(1 + \cos^2 x)$.

48. $\int \frac{\sin 2x}{1 + \sin x} dx$

Use the double angle formula to write $\int \frac{\sin 2x}{1 + \sin x} dx = \int \frac{2 \sin x \cos x}{1 + \sin x} dx$, substitute $u = \sin x$ so the resulting integral becomes $\int \frac{2u \cos x}{1 + u} du$ with $u = \sin x$.

Now $\frac{2u}{1+u} = 2 - \frac{2}{1+u}$ which is the derivative of $2u - 2 \ln(1 + u)$. Hence

$$\int \frac{\sin 2x}{1 + \sin x} dx = 2 \sin x - 2 \ln(1 + \sin x) + C.$$

49. $\int_0^1 z \sqrt{1 - z^2} dz.$

Substitute $u = 1 - z^2$, $du = -2z dz$ and get

$$\int_0^1 z \sqrt{1 - z^2} dz = \int_1^0 \left(-\frac{1}{2}\right) u^{1/2} du = \int_0^1 \frac{1}{2} u^{1/2} du = \frac{1}{3}.$$

50. Substitute $u = \ln(2x)$, $du = \frac{1}{x} dx$ and get

$$\int_1^2 \frac{\ln 2x}{x} dx = \int_{\ln 2}^{\ln 4} u du = \frac{(\ln 4)^2 - (\ln 2)^2}{2}.$$

51. $\int_{\xi=0}^{\sqrt{2}} \xi(1+2\xi^2)^{10} d\xi$. Substitute $u = 1 + 2\xi^2$, $du = 4\xi d\xi$ and the integral becomes $\int_1^5 \frac{1}{4}u^{10}du = \frac{5^{11}-1}{44}$.

52. $\int_2^3 \sin \rho (\cos 2\rho)^4 d\rho$.

Write $\cos(2\rho) = \cos^2 \rho - \sin^2 \rho = 2 \cos^2 \rho - 1$ and then the substitution $u = \cos \rho$, $du = -\sin \rho d\rho$ yields

$$\int_2^3 \sin \rho (\cos 2\rho)^4 d\rho = - \int_{\cos 2}^{\cos 3} (2u^2 - 1)^4 du$$

which is the integral of a polynomial and can be computed.

52-alt. $\int_2^3 \sin 2\rho (\cos 2\rho)^4 d\rho$. This is easier – use the substitution $u = \cos 2\rho$.

53. $\int \alpha e^{-\alpha^2} d\alpha$. Use $u = -\alpha^2$.

54. $\int \frac{e^{\frac{1}{t}}}{t^2} dt$. Use $u = 1/t$

55. (i) If $y = \sin \theta$, express $\sin \theta$, $\cos \theta$, and $\tan \theta$ in terms of y when $0 \leq \theta < \pi/2$.

$$\sin \theta = y, \cos \theta = (1 - y^2)^{1/2}, \tan \theta = \frac{y}{\sqrt{1-y^2}}.$$

(ii) If $y = \sin \theta$, express $\sin \theta$, $\cos \theta$, and $\tan \theta$ in terms of y when $\pi/2 < \theta \leq \pi$.

$$\sin \theta = y, \cos \theta = -(1 - y^2)^{1/2}, \tan \theta = -\frac{y}{\sqrt{1-y^2}}.$$

(iii) If $y = \sin \theta$, express $\sin \theta$, $\cos \theta$, and $\tan \theta$ in terms of y when $-\pi/2 < \theta < 0$.

$$\sin \theta = y, \cos \theta = (1 - y^2)^{1/2}, \tan \theta = \frac{y}{\sqrt{1-y^2}} \text{ (now } y \text{ is negative!).}$$

(iv) Evaluate $\int \frac{dy}{\sqrt{1-y^2}}$ using the substitution $y = \sin \theta$, but give the final answer in terms of y .

The problem asks for the antiderivative(s) of $f(y) = \frac{dy}{\sqrt{1-y^2}}$ in the interval $(-1, 1)$. The integral is equal to $\int \frac{\cos \theta}{\cos \theta} d\theta$ i.e. equal to θ where $\theta = \arcsin y$.

56. (a) Simplify $\cos(\sin^{-1}(x))$. Here $\sin^{-1}(x)$ was supposed to be written as $\arcsin(x)$.

$\arcsin x$ is defined on $(-1, 1)$ and the values of $\arcsin x$ are between $(-\pi/2, \pi/2)$. $\cos \alpha \geq 0$ for $-\pi/2 \leq \alpha \leq \pi/2$. Hence on this interval $\cos \alpha = \sqrt{1 - \sin^2 \alpha}$. Thus

$$\cos(\arcsin x) = \sqrt{1 - [\sin(\arcsin x)]^2} = \sqrt{1 - x^2}.$$

(b) Simplify $\tan \left\{ \arcsin \frac{\ln \frac{1}{4}}{\ln 16} \right\}$.

First simplify $\frac{\ln \frac{1}{4}}{\ln 16} = \frac{\ln 2^{-2}}{\ln 2^4} = \frac{(-2) \ln 2}{4 \ln 2} = \frac{-2}{4} = -\frac{1}{2}$.

Now $\arcsin(-\frac{1}{2})$ is the angle in $(-\pi/2, \pi/2)$ whose sin is $-1/2$. I.e. $\arcsin(-\frac{1}{2}) = -\pi/6$.

Finally $\tan(-\pi/6) = \frac{\sin(-\pi/6)}{\cos(-\pi/6)} = \frac{-\sin(\pi/6)}{+\cos(\pi/6)} = -\frac{1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$.

(c) Simplify $\sin(2 \arctan a)$.

We want to use $\tan(\arctan a) = a$.

Let's write $\sin(2x) = 2 \sin x \cos x = 2 \tan x \cos^2 x$.

Now $\tan^2 x = \frac{1 - \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1$ and therefore $\cos^2 x = \frac{1}{1 + \tan^2 x}$.

Apply this for $x = \arctan a$ to get

$$\sin(2 \arctan a) = 2 \tan(\arctan a) \frac{1}{1 + \tan^2(\arctan a)} = \frac{2a}{1 + a^2}$$

57. The graph of $y = f(x) = \arcsin(\sin(x))$, for $-2\pi \leq x \leq +2\pi$.

Hint: What is $\arcsin(w)$?

Answer: it is the angle θ between $-\pi/2$ and $\pi/2$ which makes $\sin \theta = w$.

Thus $\arcsin(\sin \theta) = \theta$ if $-\pi/2 \leq \theta \leq \pi/2$.

Now let us examine what happens if $\pi/2 < \theta < 3\pi/2$. Draw the graph of the sin and notice (or remember) the symmetry with respect to the line $x = \pi/2$. That is, we have $\sin(\frac{\pi}{2} + \alpha) = \sin(\frac{\pi}{2} - \alpha)$ which can also be expressed as $\sin \theta = \sin(\pi - \theta)$. Therefore if θ lies in the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$ then $\pi - \theta \in [-\pi/2, \pi/2]$ and therefore $\arcsin(\sin \theta) = \arcsin(\sin(\pi - \theta)) = \pi - \theta$ for $\pi/2 < \theta \leq 3\pi/2$.

Summary:

$$\arcsin(\sin \theta) = \begin{cases} \theta & \text{if } -\pi/2 \leq \theta \leq \pi/2 \\ \pi - \theta & \text{if } \pi/2 \leq \theta \leq 3\pi/2 \end{cases}$$

This allows you to draw the graph of $\arcsin(\sin x)$ for $-\pi/2 \leq x \leq 3\pi/2$. Now argue that $\arcsin(\sin x) = \arcsin(x + 2\pi)$ for all x and draw the full graph.

63. $\int_{-1/2}^{1/2} (4-x^2)^{-1/2} dx = \frac{1}{2} \int_{-1/2}^{1/2} (1-(\frac{x}{2})^2)^{-1/2} dx$. Substitute $u = x/2$, $du = \frac{1}{2} dx$ to see that this integral is equal to

$$\int_{-1/4}^{1/4} (1-u^2)^{-1/2} du = \arcsin(1/4) - \arcsin(-1/4)$$

(which is equal to $2 \arcsin(1/4)$).

64. $\int_{-1}^1 (4-x^2)^{-1/2} dx = \frac{1}{2} \int_{-1}^1 (1-(x/2)^2)^{-1/2} dx$. As above we see that this is equal to $\int_{-1/2}^{1/2} (1-u^2)^{-1/2} du = \arcsin(1/2) - \arcsin(-1/2) = \pi/6 - (-\pi/6) = \pi/3$.

67. $\int \frac{dx}{x^2+a^2} dx = \frac{1}{a^2} \int \frac{dx}{(x/a)^2+1}$. Substitute $u = x/a$, $du = \frac{1}{a} dx$ and get the answer $\frac{1}{a} \arctan(\frac{x}{a})$.

70. $\int_a^{a\sqrt{3}} \frac{dx}{x^2+a^2} = \int_a^{a\sqrt{3}} \frac{1}{a^2} \frac{dx}{(x/a)^2+1}$. Substitute $u = x/a$ and get the integral $a^{-1} \int_1^{\sqrt{3}} \frac{du}{u^2+1} du = a^{-1}(\arctan(\sqrt{3}) - \arctan 1) = a^{-1}(\pi/3 - \pi/4) = \frac{\pi}{12a}$

75. $\int \sin^n x dx = \int \sin^{n-1} x \sin x dx$ and we integrate by parts to see that the last term is

$$\sin^{n-1} x (-\cos x) - \int (n-1) \sin^{n-2} x \cos x (-\cos x) dx$$

and since $\cos^2 x = 1 - \sin^2 x$ this is

$$\begin{aligned} &= \sin^{n-1}x(-\cos x) + \int (n-1)\sin^{n-2}x(1-\sin^2 x)dx \\ &= \sin^{n-1}x(-\cos x) + (n-1)\int \sin^{n-2}x dx - (n-1)\int \sin^n x dx \end{aligned}$$

Hence $n \int \sin^n x dx = -\sin^{n-1}x \cos x + (n-1) \int \sin^{n-2}x dx$ which gives the answer.

79. Use integration by parts.

Or, alternatively, observe that by the product rule (for $x > 0$)

$$\begin{aligned} \frac{d}{dx} \left(x^{m+1} \ln x \right) &= (m+1)x^m(\ln x) + x^{m+1}n(\ln x)^{n-1}x^{-1} \\ &= x^m \left((m+1)(\ln x)^n + n(\ln x)^{n-1} \right). \end{aligned}$$

Divide by $m+1$, take antiderivatives and see

$$\frac{x^{m+1}}{m+1} \ln x = \int x^m (\ln x)^n dx + \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx$$

which yields the assertion.

83. Substitute $u = \ln x$, $du = \frac{1}{x} dx$ to get $\int \frac{\ln x}{x} dx = \int u du$ and the result is $(\ln x)^2/2$.

84.

$$\frac{d}{dx} \left(\frac{1}{n-1} \tan^{n-1} x \right) = \tan^{n-2} x \frac{1}{\cos^2 x}.$$

But $\frac{1}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \tan^2 x + 1$ and thus we obtain that the derivative of $\frac{1}{n-1} \tan^{n-1} x$ equals $\tan^{n-2} x (1 + \tan^2 x)$, hence $\tan^{n-2} x + \tan^n x$. This implies

$$\frac{1}{n-1} \tan^{n-1} x = \int \tan^{n-2} x dx + \int \tan^n x dx$$