94. Factor $x^2 + 6x + 8 = (x+4)(x+2)$. We do the partial fraction decomposition :

$$\frac{1}{x^2+6x+8} = \frac{1}{(x+4)(x+2)} = \frac{A}{x+4} + \frac{B}{x+2}.$$

Now $A = \lim_{x\to -4} \frac{1}{x+2} = -\frac{1}{2}$ and $B = \lim_{x\to -2} \frac{1}{x+4} = \frac{1}{2}$. Alternatively you can compute $\frac{A}{x+4} + \frac{B}{x+2} = \frac{(A+B)x+2A+4B}{(x+4)(x_2)}$ and then get A, B from solving $A+B=0, 2A+4B=1$.

For the integral we get

$$\int \frac{1}{x^2+6x+8}dx = \int \frac{-1/2}{x+4}dx + \int \frac{1/2}{x+2}dx$$

which gives as an answer $-\frac{1}{2}\ln|x+4| + \frac{1}{2}\ln|x+2| + C$.

95. Write $x^2+6x+10 = (x+3)^2+1$ so that

$$\int \frac{1}{x^2+6x+10}dx = \arctan(x+3) + C.$$

96.

$$\int \frac{dx}{5x^2+20x+25} = \frac{1}{5}\int \frac{dx}{x^2+4x+5}dx = \frac{1}{5}\int \frac{dx}{(x+2)^2+1}$$

which gives $\frac{1}{5}\arctan(x+2)+C$ as the answer.

100. Use $\frac{x^4-1}{x^2+1} = x^2-1$.

103. First substitute $u = x^2-1$, $du = 2x\,dx$ so that $\int \frac{x^5}{x^2-1}dx = \int \frac{(1+u)^2}{u}du = \int [\frac{1}{u} + 2 + u\,du$ with $u = x^2-1$. So an answer is $\ln(x^2-1) + 2(x^2-1) + \frac{1}{2}(x^2-1)^2$.

104. Use the substitution $u = e^x$, $du = e^x dx$. Then

$$\int \frac{e^{3x}}{e^{4x}-1}dx = \int \frac{e^{2x}}{e^{4x}-1}\,e^x dx = \int \frac{u^2}{u^4-1}\,du \text{ with } u = e^x.$$

Now we do partial fractions:

$$\frac{u^2}{u^4-1} = \frac{u^2}{(u-1)(u+1)(u^2+1)} = \frac{A}{u-1} + \frac{B}{u+1} + \frac{Cu+D}{u^2+1}.$$

The right hand side is equal to

$$\frac{A(u^2+1)(u+1) + B(u^2+1)(u-1) + (Cu+D)(u-1)(u+1)}{(u-1)(u+1)(u^2+1)} =$$

$$\frac{(A+B+C)u^3+(A-B+D)u^2+(A+B-C)u+(A-B-D)}{(u-1)(u+1)(u^2+1)}$$

and thus we have

$u^2 = (A+B+C)u^3+(A-B+D)u^2+(A+B-C)u+(A-B-D)$

for all u which is satisfied if

$$\begin{aligned} A+B+C &= 0 \\ A-B+D &= 1 \\ A+B-C &= 0 \\ A-B-D &= 0 \end{aligned}$$

Solving this system gives $A = 1/4$, $B = -1/4$, $C = 0$, $D = 1/2$. Thus

$$\begin{aligned} \int \frac{u^2}{u^4-1}\,du &= \frac{1}{4}\int \frac{du}{u-1} - \frac{1}{4}\int \frac{du}{u+1}\,du + \frac{1}{2}\int \frac{du}{u^2+1} \\ &= \frac{1}{4}\ln|u-1| - \frac{1}{4}\ln|u+1| + \frac{1}{2}\arctan u + \mathit{constant}. \end{aligned}$$

Thus for the original integral

$$\int \frac{e^{3x}}{e^{4x}-1}dx = \frac{1}{4}\ln|e^x-1| - \frac{1}{4}\ln|e^x+1| + \frac{1}{2}\arctan e^x + \mathit{constant}.$$

105. Substitute $u = e^x$, $du = e^x\,dx$ so that

$$\int \frac{e^x}{(1+e^{2x})^{1/2}}\,dx = \int \frac{1}{(1+u^2)^{1/2}}du = \ln(u+\sqrt{u^2+1})$$

with $u = e^x$. here we used $\sinh^{-1}$. Thus the answer is $\ln(e^x + \sqrt{e^{2x}+1})$.

111. Write

$$f(x) := \frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

The right hand side is equal to

$$\frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$
$$= \frac{(A+B+C)x^2 + (-5A-4B-3C)x + (6A+3B+2C)}{(x-1)(x-2)(x-3)}$$

and this is equal to $\frac{x^2+1}{(x-1)(x-2)(x-3)}$ if

$$A+B+C = 1$$
$$-5A-4B-3C = 0$$
$$6A+3B+2C = 1$$

Solve this system of equations to get $C = 5$, $B = -5$, $A = 1$. This is perhaps not too hard but requires a computation. *Alternatively* use the Heaviside trick which is applicable since the denominator has only simple roots.

We compute

$$A = \lim_{x\to 1}(x-1)f(x) = \lim_{x\to 1}\frac{x^2+1}{(x-2)(x-3)} = 1$$
$$B = \lim_{x\to 2}(x-2)f(x) = \lim_{x\to 2}\frac{x^2+1}{(x-1)(x-3)} = -5$$
$$C = \lim_{x\to 3}(x-3)f(x) = \lim_{x\to 3}\frac{x^2+1}{(x-1)(x-2)} = 5$$

It is of course good to check your answers by applying both methods.

Back to the integral:

$$\int \frac{x^2+1}{(x-1)(x-2)9x-3)}\,dx = \int \Big[\frac{1}{x-1} + \frac{-5}{x-2} + \frac{5}{x-3}\Big]$$
$$= \ln|x-1| - 5\ln|x-2| + 5\ln|x-3| \,+\, \text{constant}.$$

Note that it would not be good to use the C for an arbitrary constant here, because the letter C in this problem was already used.

117. Use integration by parts:

$$\int_0^a x\sin x\,dx = \Big[x(-\cos x)\Big]_0^a - \int_0^a (-\cos x)\,dx = -a\cos a + \sin a.$$

118. Use one more integration by parts to reduce to the previous problem.

119. $\int \frac{x}{\sqrt{x^2-1}}\,dx$. Use the substitution $u = x^2 - 1$.

120. Use the substitution $u = 1 - x^2$, $du = -2x\,dx$, and get

$$\int_{1/4}^{1/3} \frac{x}{\sqrt{1-x^2}}\,dx = \int_{15/16}^{8/9} u^{-1/2}(-\frac{1}{2})\,du$$
$$= \int_{8/9}^{15/16} \frac{1}{2} u^{-1/2}\,du = \Big[u^{1/2}\Big]_{15/16}^{8/9} = \frac{2\sqrt{2}}{3} - \frac{\sqrt{15}}{4}.$$

121. *First method:*

In the integral $\int_3^4 \frac{dx}{x\sqrt{x^2-1}}$ we substitute $u = (x^2-1)^{1/2}$ so that $du = x(x^2-1)^{-1/2}dx$ and we have $x^2 = 1 + u^2$. Then

$$\int_3^4 \frac{dx}{x\sqrt{x^2-1}} = \int_3^4 \frac{1}{x^2}\,\frac{x}{\sqrt{x^2-1}}\,dx = \int_{(3^2-1)^{1/2}}^{(4^2-1)^{1/2}} \frac{1}{1+u^2}\,du$$

which is $\arctan(\sqrt{15}) - \arctan(2\sqrt{2})$.

Second method: Substitute $v = 1/x$, $dv = -1/x^2 dx$. Then

$$\int_3^4 \frac{dx}{x\sqrt{x^2-1}} = \int_3^4 \frac{x}{x^2\sqrt{x^2-1}}dx = -\int_{1/3}^{1/4} \frac{v^{-1}}{\sqrt{v^{-2}-1}}dv$$
$$= \int_{1/4}^{1/3} \frac{1}{\sqrt{1-v^2}}dv = \arccos(1/3) - \arccos(1/4).$$

Verify that the two answers are consistent by showing that for $x > 1$ we have the relation $\arccos\frac{1}{x} = \arctan(\sqrt{x^2-1})$.

122. Write $x^2 + 2x + 17 = (x+1)^2 + 4^2$ and

$$\int \frac{x\,dx}{x^2+2x+17}dx = \frac{1}{2}\int \frac{2(x+1)}{(x+1)^2+4^2}\,dx - \int \frac{1}{(x+1)^2+4^2}\,dx$$

which yields $\frac{1}{2}\ln((x+1)^2+4^2) - \frac{1}{4}\arctan(\frac{x+1}{4}) + C$.

123. We use (for $x > 6$) the substitution $x = 6\cosh u$, $dx = 6\sinh u$ and compute (using $\cosh^2 u = 1 + \sinh^2 u$)

$$\int \frac{x^4}{\sqrt{x^2-36}}\,dx = \int \frac{6^4(\cosh u)^4}{\sqrt{6^2(\cosh^2 u - 1)}} 6\sinh u\,du = 6^4 \int (\cosh u)^4 du$$
$$= 6^4 \int \Big(\frac{e^u + e^{-u}}{2}\Big)^4 du$$

which can be solved by multiplying out the fourth powers and then integrating the resulting term

$$6^4 \cdot 2^{-4} \int (e^{4u} + 4e^{2u} + 6 + 4e^{-2u} + e^{-4u})\, du.$$

The resubstitution $u = \cosh^{-1}(x/6)$ then yields an answer.

123-modified. Let's do instead $\int \frac{x^4}{\sqrt{36-x^2}}\, dx$ (for $|x| < 6$). Then we substitute $x = 6\sin t$, $dx = 6\cos t$ and get the integral $6^4 \int \frac{\sin^4 t}{\cos t} \cos t\, dt = 6^4 \int \sin^4 t\, dt$ which can be solved by several applications of the double angle formulas.

124. Here we compute $\frac{x^4}{x^2-36} = \frac{x^4-6^4}{x^2-6^2} + \frac{6^4}{x^2-6^2} = x^2 + 36 + \frac{6^4}{(x-6)(x+6)}$. Then write $\frac{1}{(x-6)(x+6)} = \frac{A}{x-6} + \frac{B}{x+6}$ and determine A, B..

125. Use $\frac{x^4}{36-x^2} = -\frac{x^4}{x^2-36}$ and apply 124. How interesting is that?

126. Do partial fractions

$$\frac{x^2+1}{x^4-x^2} = \frac{x^2+1}{x^2(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}.$$

128. Substitute, for $x > \sqrt{3}$, $x = \sqrt{3}\cosh t$, $dx = \sqrt{3}\sinh t\, dt$, and get $\int \frac{dx}{(x^2-3)^{1/2}} = \int 1\, dt = t + C = \cosh^{-1}(\frac{x}{\sqrt{3}})$.

128-modified For $\int \frac{dx}{\sqrt{3-x^2}}$, $|x| < \sqrt{3}$ you may substitute $x = \sqrt{3}\sin t$, $dx = \sqrt{3}\cos t$ and get $\int \frac{dx}{(3-x^2)^{1/2}} = \int 1\, dt = t + C = \arcsin(\frac{x}{\sqrt{3}})$. Of course, you could simply reduce to the formula for the derivative of arcsin on page 9.

What happens of you substitute $x = \sqrt{3}\cos t$?

129. Split into $\int xe^x dx + \int e^x \cos x\, dx$ and use problems 73/74.

130. Answer: $e^x + x\ln x - x + C$.

131. Write $\frac{3x^2+2x-2}{x^3-1} = \frac{3x^2+2x-2}{(x-1)(x^2+x+1)}$ and

$$x^2 + x + 1 = x^2 + x + \frac{1}{4} + \frac{3}{4} = (x + \frac{1}{2})^2 + \frac{3}{4}.$$

We do the partial fraction decomposition and write

$$\frac{3x^2+2x-2}{(x-1)((x+\frac{1}{2})^2+\frac{3}{4})} = \frac{A}{x-1} + \frac{Bx+C}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

and we need to determine A, B and C. The right hand side is

$$\frac{A\big((x+\frac{1}{2})^2+\frac{3}{4}\big)+(Bx+C)(x-1)}{(x-1)((x+\frac{1}{2})^2+\frac{3}{4})} = \frac{(A+B)x^2+(A-B+C)x+A-C}{(x-1)((x+\frac{1}{2})^2+\frac{3}{4})}$$

and since the numerator must be equal to $3x^2+2x-2$ we need to solve $A+B=3$, $A-B+C=2$, $A-C=-2$. I get $A=1$, $B=2$, $C=3$.

Hence

$$\int \frac{3x^2+2x-2}{x^3-1}\,dx = \int \frac{1}{x-1}dx + \int \frac{2x+3}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$$

Now $\int \frac{1}{x-1}dx = \ln|x-1| + C$ and

$$\int \frac{2x+3}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx = \int \frac{2(x+\frac{1}{2})}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx + 2\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$$

Compute

$$\int \frac{2(x+\frac{1}{2})}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx = \ln((x+\frac{1}{2})^2+\frac{3}{4}) + C = \ln(x^2+x+1)+C$$

Finally we use (again) $\int \frac{dx}{(x-a)^2+b^2}dx = \frac{1}{b}\arctan(\frac{x-a}{b})+C$ for $a=-\frac{1}{2}$ and $b=\frac{\sqrt{3}}{2}$ and get

$$2\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \frac{4}{\sqrt{3}}\arctan(2\frac{x+\frac{1}{2}}{\sqrt{3}}) = \frac{4}{\sqrt{3}}\arctan(\frac{2x+1}{\sqrt{3}})$$

and as a final answer I get

$$\int \frac{3x^2+2x-2}{x^3-1}\,dx = \ln|x-1| + \ln(x^2+x+1) + \frac{4}{\sqrt{3}}\arctan(\frac{2x+1}{\sqrt{3}}) + C.$$

133. Split $\frac{x}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$ and compute A, B, C.

134. Split $\frac{4}{(x-1)^3(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1}$ and compute A, B, C and D.

135. Write

$$(1-2x-x^2) = 2-(1+2x+x^2) = 2-(1+x)^2 = 2\big(1-(\frac{1+x}{\sqrt{2}})^2\big).$$

Thus, with substitution $u = (1+x)/\sqrt{2}$, $du = dx/\sqrt{2}$,

$$\int \frac{dx}{\sqrt{1-x-x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{1-(\frac{1+x}{\sqrt{2}})^2}} = \frac{\sqrt{2}}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{\sqrt{2}}{2}\arcsin(\tfrac{1+x}{\sqrt{2}}) + C.$$

137. Use integration by parts. Write $g(x) = \ln x$ and $f(x) = x^2/2$ and integrate $\int_1^e f'(x)g(x)\,dx$. Then

$$\int_1^e x\ln x\,dx = \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2}\frac{1}{x}\,dx$$
$$= \frac{e^2}{2}\ln e - \frac{1^2}{2}\ln 1 - \frac{1}{2}\int_1^e x\,dx = \frac{e^2}{2} - \frac{1}{2}\frac{e^2}{2} = \frac{e^2}{4}.$$

140. Use substitution $u = x^{1/2}$, $du = \frac{1}{2}x^{-1/2}\,dx$ and get

$$\int \arctan(x^{1/2})\,dx = \int 2u\arctan u\,du$$

(with $u = x^{1/2}$). Now by integration by parts

$$\int 2u\arctan u\,du = u^2\arctan u - \int \frac{u^2}{1+u^2}\,du$$

and

$$\int \frac{u^2}{1+u^2}\,du = \int \left(1 - \frac{1}{1+u^2}\right)du$$

which is $u - \arctan u + C$. Combining everything and putting back $u = x^{1/2}$ we get

$$\int \arctan(x^{1/2})\,dx = x\arctan(\sqrt{x}) - \sqrt{x} + \arctan(\sqrt{x}) + C\,.$$

141. We use $\cos(2x) = \cos^2 x - \sin^2 x$ and $1 = \cos^2 x + \sin^2 x$, hence $\cos^2 x = \frac{1+\cos(2x)}{2}$. Thus $\int x(\cos x)^2 dx = \int \frac{x}{2}\,dx + \frac{1}{2}\int x\cos(2x)\,dx$ and the last integral can be solved with integration by parts. The answer is $\frac{1}{8}(2x^2 + x\sin(2x) + \cos(2x)) + C$.

142. We use double angle formulas and compute

$$\int_0^\pi \sqrt{1+\cos(6w)}dw = \int_0^\pi \sqrt{1+\cos^2(3w)-\sin^2(3w)}dw$$
$$= \int_0^\pi \sqrt{2\cos^2(3w)}dw = \frac{\sqrt{2}}{3}\int_0^{3\pi}\sqrt{\cos^2(x)}dx = \frac{\sqrt{2}}{3}\int_0^{3\pi}|\cos x|dx.$$

Do not forget the absolute values! Now compute $\int_0^{3\pi}|\cos x|dx$ (the value is 6).

145. To compute the integral $\int_0^a (a^{2/3} - x^{2/3})^{3/2} dx$ we first reduce to $a = 1$ by setting $x = as$, $dx = a\, ds$. So

$$\int_0^a (a^{2/3} - x^{2/3})^{3/2} dx = \int_0^1 (a^{2/3} - a^{2/3}s^{2/3})^{3/2} a\, ds = a^2 \int_0^1 (1 - s^{2/3})^{3/2}\, ds.$$

Substitute $s = (\sin\alpha)^3$, $ds = 3\sin^2\alpha\, \cos\alpha\, d\alpha$. Then

$$\int_0^1 (1 - s^{2/3})^{3/2} ds = \int_0^{\pi/2} (1 - \sin^2\alpha)^{3/2} 3\sin^2\alpha\, \cos\alpha\, d\alpha$$

$$= 3\int_0^{\pi/2} \cos^4\alpha\, \sin^2\alpha\, d\alpha$$

Now $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1$, and using also $\sin(2\alpha) = 2\sin\alpha\cos\alpha$ we get $\cos^2\alpha = \frac{1+\cos(2\alpha)}{2}$. Thus

$$\cos^4\alpha\, \sin^2\alpha = \cos^2\alpha\, (\cos\alpha\sin\alpha)^2 = \frac{(1 + \cos(2\alpha))(\sin(2\alpha)}{8}.$$

Therefore

$$3\int_0^{\pi/2} (\cos\alpha)^4 (\sin\alpha^2) d\alpha = \frac{3}{8}\int_0^{\pi/2} (1 + \cos(2\alpha))\sin(2\alpha) d\alpha$$

Using the double angle formula for the sin again we see that the last expression is equal to

$$\frac{3}{8}\int_0^{\pi/2} \sin(2\alpha) d\alpha + \frac{3}{16}\int_0^{\pi/2} \sin(4\alpha) d\alpha.$$

Making substitutions $\beta = 2\alpha$ and $\gamma = 4\alpha$ in these integrals we get

$$\frac{3}{8}\int_0^{\pi} \sin\beta\, \frac{d\beta}{2} + \frac{3}{16}\int_0^{2\pi} \sin\gamma\, \frac{d\gamma}{4}$$

and since $\int_0^\pi \sin\beta\, d\beta = 2$, $\int_0^{2\pi} \sin\gamma\, d\gamma = 0$ we get $3/16$ for the last displayed expression.

All together $\int_0^a (a^{2/3} - x^{2/3})^{3/2} dx = \frac{3a^2}{16}$.

147. The area is *four times* the integral $\int_0^1 (x^4 - x^6)^{1/2}\, dx$ and this integral can also be written as $\int_0^1 x^2(1 - x^2)^{1/2}\, dx$. Set $x = \sin t$, $dx = \cos t\, dt$. Then

$$\int_0^1 x^2(1 - x^2)^{1/2}\, dx = \int_0^{\pi/2} \sin^2 t(1 - \sin^2 t)^{1/2} \cos t\, dt$$

$$= \int_0^{\pi/2} \sin^2 t \cos^2 t\, dt = \frac{1}{4}\int_0^{\pi/2} \sin(2t)\, dt = \frac{1}{8}.$$

148. The area of the tail is *twice* the integral $\int_0^1 x\sqrt{\frac{1-x}{1+x}}dx$. Following the hint write $x\sqrt{\frac{1-x}{1+x}} = \frac{x}{1+x}\sqrt{1-x^2}$. We make the substituition $x = \sin\theta$, $dx = \cos\theta$ and get

$$\int_0^1 \frac{x}{1+x}\sqrt{1-x^2}\,dx = \int_0^{\pi/2} \frac{\sin\theta}{1+\sin\theta}\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta.$$

We need to simplify the integrand; in particular we want to get rid of the trigonometric functions in the denominator:

$$\begin{aligned}\frac{\sin\theta}{1+\sin\theta}\sqrt{1-\sin^2\theta}\cos\theta &= \frac{\sin\theta(\cos\theta)^2}{1+\sin\theta}\\ &= \frac{\sin\theta(\cos\theta)^2(1-\sin\theta)}{1-\sin^2\theta} = \sin\theta - \sin^2\theta\,.\end{aligned}$$

Now compute (using $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$ from the double angle formulas)

$$\begin{aligned}\int_0^{\pi/2}(\sin\theta - \sin^2\theta)\,d\theta &= \int_0^{\pi/2}\sin\theta\,d\theta - \int_0^{\pi/2}\sin^2\theta\,d\theta\\ &= 1 - \int_0^{\pi/2}\frac{1-\cos(2\theta)}{2}d\theta = 1 - \frac{\pi}{4}\,.\end{aligned}$$

Thus, the area of the tail is $(2 - \pi/2)$.

149. The cross sections for fixed x are disks with radius $y = y(x)$, they have area $\pi[y(x)]^2 = \pi\frac{x^2}{x^2+25}$. The volume is

$$\int \pi[y(x)]^2 dx \;=\; \int_5^{10}\pi\frac{x^2}{x^2+25}\,dx.$$

Now $\frac{x^2}{x^2+25} = 1 - \frac{25}{x^2+25} = 1 - \frac{1}{1+(x/5)^2}$ and we compute

$$\int_5^{10}\frac{1}{1+(x/5)^2}dx = \int_1^2 \frac{1}{1+u^2}5\,du = 5(\arctan 2 - \arctan 1)\,.$$

Thus the volume is

$$\pi\Big(\int_5^{10} 1\,dx - \int_5^{10}\frac{1}{1+(x/5)^2}dx\Big) = \pi(5 - 5\arctan 2 + 5\arctan 1)$$

which is equal to $\pi(5 - 5\arctan 2 + 5\pi/4)$.