H/W # 4

Problems: 159, 161, 191a & 2(iv), 3(iii) from the packet

159.) Compute $T_0^a f(x)$, $T_1^a f(x)$ & $T_2^a f(x)$
where $f(x) = \ln(x)$ & $a = 1, e^2$

Make a chart!

n	$\frac{d^n}{dx^n}\ln(x)$	$\frac{d^n}{dx^n}\ln(x)\Big\vert_1$	$\frac{d^n}{dx^n}\ln(x)\Big\vert_{e^2}$
0	$\ln(x)$	0	2
1	$1/x$	1	$1/e^2$
2	$-1/x^2$	-1	$-1/e^4$

So,

$$T_0^1 \ln(x) = 0$$

$$T_1^1 \ln(x) = 0 + (x-1) = (x-1)$$

$$T_2^1 \ln(x) = 0 + (x-1) - \frac{1(x-1)^2}{2!} = (x-1) - \frac{1}{2}(x-1)^2$$

$$T_0^{e^2} \ln(x) = 2$$

$$T_1^{e^2} \ln(x) = 2 + \frac{1}{e^2}(x-e^2)$$

$$T_2^{e^2} \ln(x) = 2 + \frac{1}{e^2}(x-e^2) - \frac{1}{e^4}\frac{(x-e^2)^2}{2!}$$

2(iv) Approach using integration by parts:

$$I = \int_0^x \sqrt{a^2+t^2}\,dt$$

$$u = \sqrt{a^2+t^2} \quad dv = 1$$

$$du = \frac{t}{\sqrt{a^2+t^2}} \quad v = t$$

$$I = t\sqrt{a^2+t^2}\Big|_0^x - \int_0^x \frac{t^2}{\sqrt{a^2+t^2}}\,dt$$

$$I = x\sqrt{a^2+x^2} - \left[\int_0^x \frac{(a^2+t^2)}{\sqrt{a^2+t^2}} - \frac{a^2}{\sqrt{a^2+t^2}}\right]$$

$$I = x\sqrt{a^2+x^2} - I + a^2\int_0^x \frac{1}{\sqrt{a^2+t^2}}\,dt$$

$$2I = x\sqrt{a^2+x^2} + a^2 \sinh^{-1}(t/a) \quad \text{(from packet)}$$

$$I = \frac{1}{2}\left[x\sqrt{a^2+x^2} + a^2\sinh^{-1}(t/a)\right]$$

3(iii) Invert $\tanh(x)$

~~[illegible]~~ $\tanh(x) = \dfrac{e^x+e^{-x}}{e^x-e^{-x}} = y$

Now, solve for e^x

$$\Longrightarrow \frac{e^{2x}+1}{e^{2x}-1} = y$$

$$\Longrightarrow e^{2x}+1 = y(e^{2x}-1)$$

$$\Longrightarrow e^{2x}(1-y) = -(y+1)$$

$$\Longrightarrow e^{2x} = \left(\frac{y+1}{y-1}\right)$$

$$\Longrightarrow e^x = \left(\frac{y+1}{y-1}\right)^{1/2}$$

Now take the ln of both sides

$$x = \frac{1}{2}\ln\left(\frac{y+1}{y-1}\right) = \frac{1}{2}\left[\ln(y+1) - \ln(y-1)\right] ✓$$

161.) Compute $T_0^a f(x)$, $T_1^a f(x)$ + $T_2^a f(x)$
where $f(x) = \sin(2x)$, $a = 0, \pi/4$

Chart:

n	$\frac{d^n}{dx^n}\sin(2x)$	$\frac{d^n}{dx^n}\sin(2x)\big\vert_0$	$\frac{d^n}{dx^n}\sin(2x)\big\vert_{\pi/4}$
0	$\sin(2x)$	0	1
1	$2\cos(2x)$	2	0
2	$-4\sin(2x)$	0	-4

Note: $\sin(2x)$ centered @ $a=0$ is an <u>odd</u> function. $\sin(2x)$ centered @ $\pi/4$ is <u>even</u>.

So,

$T_0^0 \sin(2x) = 0$

$T_1^0 \sin(2x) = 0 + 2x = 2x$

$T_2^0 \sin(2x) = 0 + 2x + 0 = 2x$

$T_0^{\pi/4} \sin(2x) = 1$

$T_1^{\pi/4} \sin(2x) = 1 + 0 = 1$

$T_2^{\pi/4} \sin(2x) = 1 + 0 - \frac{4(x-\pi/4)^2}{2!} = 1 - 2(x-\pi/4)^2$

191 a.) if $f(x) = (8+x)^{1/3}$ find $T_2^0 f(x)$

n	$\frac{d^n}{dx^n} f(x)$	$\frac{d^n}{dx^n} f(x)\big\vert_0$
0	$(8+x)^{1/3}$	2
1	$\frac{1}{3}(8+x)^{-2/3}$	$\frac{1}{3}\cdot\frac{1}{2^2}$
2	$-\frac{2}{9}(8+x)^{-5/3}$	$-\frac{2}{9}\frac{1}{2^5}$
3	$\frac{10}{27}(8+x)^{-8/3}$	$\frac{10}{27}\frac{1}{2^8}$

Note: if $0 \le c \le x$, $(8+c)^{-8/3}$ is ~~largest~~ largest when $c = 0$.

$$T_2^0 f(x) = 2 + \left(\frac{1}{3}\cdot\frac{1}{2^2}\right)x - \left(\frac{2}{9}\frac{1}{2^5}\right)\frac{x^2}{2!}$$

$$R_2^0 f(x) = \frac{10}{27}(8+c)^{-8/3}\frac{x^3}{3!} \quad \text{for some } 0 \le c \le x$$

Tailors thm says:

$$\text{error} = \left|(8+x)^{1/3} - T_2^0(8+x)^{1/3}\right| = \left|R_2^0(8+x)^{1/3}\right| = \left|\frac{10}{27}\frac{1}{3!}(8+c)^{-8/3}x^3\right|$$

if $x = 1$

$$\text{error} = \left|\frac{10}{27}\frac{1}{3!}(8+c)^{-8/3}1^3\right| \le \left|\frac{10}{27}\frac{1}{3!}8^{-8/3}\right| = \left|\frac{10}{27\cdot 3!\cdot 2^8}\right|$$