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We first want to find

$$f(t) = te^{-t} = T_3 f(t) + R_3 f(t)$$

where

$$T_3 f(t) = f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3$$

and

$$R_3 f(t) = \frac{f^{(4)}(\xi)}{4!}t^4$$

Remark. The reason we study $f(t) = te^{-t}$ because the integral we want to approximate is

$$I = \int_0^{0.1} x^2 e^{-x^2} dx$$

The integrand is $x^2 e^{-x^2} = f(x^2)$. (Let $t = x^2$).
The x is between $0 \le x \le 0.1$, $\Rightarrow$ $0 \le x^2 \le 0.01 \Rightarrow 0 \le t \le 0.01$

We compute $T_3 f(t)$.

$$f(t) = te^{-t} \qquad f(0) = 0$$

$$f'(t) = e^{-t} - te^{-t} \qquad f'(0) = 1$$

$$f''(t) = -e^{-t} - e^{-t} + te^{-t} = -2e^{-t} + te^{-t} \qquad f''(0) = -2$$

$$f'''(t) = 2e^{-t} + e^{-t} - te^{-t} = 3e^{-t} - te^{-t} \qquad f'''(0) = 3$$

$$f^{(4)}(t) = -3e^{-t} - e^{-t} + te^{-t} = -4e^{-t} + te^{-t} = (-4+t)e^{-t}$$

Hence $T_3 f(t) = t - t^2 + \frac{1}{2}t^3$

and

$$R_3 f(t) = \frac{f^{(4)}(\xi)}{24}t^4 = \frac{(-4+\xi)e^{-\xi}}{24}t^4$$

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Now, we find an upper bound of $R_3f(t)$

$$|R_3f(t)| = \left|\frac{(-4+\xi)e^{-\xi}}{24}t^4\right|$$

$$\le \frac{|-4+\xi|\cdot|e^{-\xi}|}{24}|t^4|$$

Since $\xi \ge 0$ (ξ is between $0, t$, $\overset{0.01}{\ge} t \ge 0$)

$$|-4+\xi| = 4-\xi \le 4$$

$$|e^{-\xi}| = e^{-\xi} \le 1 \qquad \xi^4 \le t^4$$

Hence

$$|R_3f(t)| \le \frac{4\cdot 1}{24}\cdot t^4 = \frac{t^4}{6}$$

To sum up, we find.

$$f(t) = T_3f(t) + R_3f(t)$$

$$T_3f(t) = t - t^2 + \frac{1}{2}t^3$$

and $\quad |R_3f(t)| \le \dfrac{t^4}{6}$

Now we study original integral

$$I = \int_0^{0.1} x^2 e^{-x^2}dx = \int_0^{0.1} f(x^2)\,dx$$

$$= \int_0^{0.1} T_3f(x^2) + R_3f(x^2)dx$$

$$= \int_0^{0.1} T_3f(x^2)\,dx + \int_0^{0.1} R_3f(x^2)\,dx$$

Note $\int_0^{0.1} T_3f(x^2)\,dx$ is the approximation to integral, while page 3. 256.

$\int_0^{0.1} R_3 f(x^2)dx$ is the error of above approximation.

Do the computation:

$$\int_0^{0.1} T_3 f(x^2)dx = \int_0^{0.1} x^2 - (x^2)^2 + \frac{1}{2}(x^2)^3 \, dx$$

$$= \left(\frac{x^3}{3} - \frac{x^5}{5} + \frac{1}{2} \times \frac{x^7}{7}\right)\Big|_0^{0.1}$$

$$= \frac{0.1^3}{3} - \frac{0.1^5}{5} + \frac{0.1^7}{14} = \cdots$$

Error $\left|\int_0^{0.1} R_3 f(x^2)dx\right| \le \int_0^{0.1} \frac{(x^2)^4}{6} dx = \frac{1}{6}\frac{x^9}{9}\Big|_0^{0.1}$

$$= \frac{0.1^9}{54}$$

Finally, we get

$I = \int_0^{0.1} x^2 e^{-x^2} dx$ is approximately $\leftarrow$ with error less than $\frac{0.1^9}{54}$.

Done!

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257. We first study $g(t) = (1+t)^{\frac{1}{2}}$ $t > 0$

$$g(t) = T_n g(t) + R_n g(t)$$

where $T_n g(t) = g(0) + g'(0)t + \cdots + \frac{g^{(n)}(0)}{n!} t^n$

$$R_n g(t) = \frac{g^{(n+1)}(\xi)}{(n+1)!} t^{n+1}$$

I don't know what the n is. But I decide to pick $n=1$ to try. If I succeed, I'm done. If I fail, I pick larger n, say $n=2,3$ to try. After some trials, I know "some" n is good!

To find $T_1 g(t)$, $R_1 g(t)$, we need to compute derivatives.

$g'(t) = \frac{1}{2}(1+t)^{-\frac{1}{2}}$ $\qquad g'(0) = \frac{1}{2}$

$g''(t) = -\frac{1}{4}(1+t)^{-\frac{3}{2}}$ $\qquad g''(0) = -\frac{1}{4}$

Hence

$$g(t) = (1 + \tfrac{1}{2}t) + \frac{(-\frac{1}{4})(1+\xi)^{-\frac{3}{2}}}{2!} t^2$$

The $R_1 g(t) = \dfrac{(-\frac{1}{4})(1+\xi)^{-\frac{3}{2}}}{2!} t^2$

$$|R_1 g(t)| = \left| \frac{(-\frac{1}{4})(1+\xi)^{-\frac{3}{2}}}{2!} t^2 \right| \leq \frac{1}{8} \cdot 1 \cdot t^2 = \frac{t^2}{8}$$

The last inequality above is because

$$|(1+\xi)^{-\frac{3}{2}}| = \frac{1}{|(1+\xi)^{\frac{3}{2}}|} \leq \frac{1}{(1+0)^{\frac{3}{2}}} = 1$$

recall $0 \leq \xi \leq t$

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Hence
$$g(t) = 1 + \frac{1}{2}t + R_1 g(t)$$
with
$$|R_1 g(t)| \le \frac{t^2}{8}$$

Do. $t = x^4$ substitution.
$$g(x^4) = 1 + \frac{1}{2}x^4 + R_1 g(x^4)$$
with
$$|R_1 g(x^4)| \le \frac{1}{8}(x^4)^2 = \frac{x^8}{8}$$

Hence
$$\int_0^{0.5} \sqrt{1+x^4}\,dx = \int_0^{0.5} g(x^4)\,dx$$
$$= \int_0^{0.5} 1 + \frac{1}{2}x^4 + R_1 g(x^4)\,dx$$
$$= \int_0^{0.5} 1 + \frac{1}{2}x^4\,dx + \int_0^{0.5} R_1 g(x^4)\,dx$$
$$= \left(x + \frac{x^5}{10}\right)\Big|_0^{0.5} + \int_0^{0.5} R_1 g(x^4)\,dx$$

$$|\text{error}| = \left|\int_0^{0.5} R_1 g(x^4)\,dx\right| \le \int_0^{0.5} |R_1 g(x^4)|\,dx$$
$$\le \int_0^{0.5} \frac{x^8}{8}\,dx = \frac{1}{8} \times \frac{x^9}{9}\Big|_0^{0.5} = \frac{1}{8} \times \frac{0.5^9}{9} = \frac{1}{72 \times 512} < 10^{-4}!$$

This tells us that the error is smaller than 10^{-4}.
"n=1" is Good!!

Hence the approximation is $\int_0^{0.5} 1 + \frac{1}{2}x^4\,dx = 0.5 + \frac{0.5^5}{10} = \ldots\ldots$

Done!

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258. $f(x) = \arctan x = T_n f(x) + R_n f(x)$

$T_n f(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n$

$R_n f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$ ξ is between $0, x$.

Hence $I = \int_0^{0.1} \arctan x \, dx = \int_0^{0.1} T_4 f(x) dx + \int_0^{0.1} R_? f(x) dx$

note $0 \le x \le 0.1$. Hence $0 \le \xi \le x \le 0.1$

To find $T_n f(x)$, $R_n f(x)$, we need

$f'(x) = \frac{1}{1+x^2}$

$f''(x) = -\frac{2x}{(1+x^2)^2}$

$f'''(x) = -2(1+x^2)^{-2} + 8x^2(1+x^2)^{-3}$.

I feel these derivatives ~~are~~ have too many terms. Hard! So I decide to try $n=2$. If I'm lucky, $n=2$ is good. If $n=2$ is not enough to make error smaller than 0.001, then I pick $n=3$, ~~to~~ try.

To estimate $\int_0^{0.1} R_2 f(x) dx$, we need an upper bound for $|R_2 f(x)|$

$$|R_2 f(x)| = \left| \frac{f'''(\xi)}{3!} x^3 \right| = \frac{\left| -2(1+\xi^2)^{-2} + 8\xi^2(1+\xi^2)^{-3} \right|}{6} |x|^3$$

$$\le \frac{|2(1+\xi^2)^{-2}| + |8\xi^2(1+\xi^2)^{-3}|}{6} |x|^3.$$

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$$|R_2 f(x)| \le \frac{2 + 8\xi^2}{6} x^3 \quad \left(\text{since } (1+\xi^2)^{-2} \le 1,\ (1+\xi^2)^{-3} \le 1 \atop 0 \le \xi \le 0.1, \right)$$

$$\le \frac{2 + 8 \times 0.1^2}{6} \cdot x^3.$$

Hence $\left|\int_0^{0.1} R_2 f(x)\,dx\right| \le \int_0^{0.1} |R_2 f(x)|\,dx$

$$\le \int_0^{0.1} \frac{2+8\times 0.1^2}{6} \cdot x^3\,dx$$

$$= \frac{2+0.08}{6} \times \frac{x^4}{4}\Big|_0^{0.1}$$

$$= \frac{2.08}{6} \times \frac{0.1^4}{4} < 0.001.$$ Good!

So $n=2$ is good enough and the approximation is

$$\int_0^{0.1} T_2 f(x)\,dx = \int_0^{0.1} f(0) + f'(0)x + \frac{f''(0)}{2}x^2\,dx$$

$$= \int_0^{0.1} 0 + x + \cancel{2}0\,dx$$

$$= \int_0^{0.1} x\,dx$$

$$= \frac{x^2}{2}\Big|_0^{0.1}$$

$$= \frac{0.1^2}{2} = 0.02.$$