Mathematics 234

A practice exam

1. You are given the curve C parametrized by

$$\mathbf{r}(t) = t\mathbf{i} + \frac{2}{3}\sqrt{2}t^{3/2}\mathbf{j} + \frac{1}{2}t^2\mathbf{k},$$

where $0 \leq t \leq 2$.

(i) Find the total length of the curve.

(ii) Find $\int_C \sqrt{|2z|}\, ds$.

(iii) Let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$. Compute the path integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

2. Assume the differentiable function of three variables, $f(x, y, z)$, satisfies

$$(*) \qquad f(P) = 6, \qquad f_x(P) = 2,\ f_y(P) = 1,\ f_z(P) = -3.$$

for a given point P.

(a) Compute the value of the directional derivative of f at P in the direction $\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} + \frac{2}{\sqrt{6}}\mathbf{k}$.

(b) Find the direction of maximal increase of f at P.

(c) Suppose you are told that there is a direction u in which the directional derivative of f at P equals 0. Can this be true? If so, find such a direction u. If not, explain why not.

(d) Suppose you are told that there is a direction u in which the directional derivative of f at P equals -20. Can this be true? If so, find such a direction u. If not, explain why not.

(e) Suppose that $P = (0, 0, 0)$ and (*) holds. Note that the curve parametrized by $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j} + 4t\mathbf{k}$ passes through the origin at time $t = 0$. Find the derivative $\frac{d}{dt}\big(f(\mathbf{r}(t))\big)$, at time $t = 0$.

3. Consider the surface defined by

$$(*) \qquad 3x + \sin(z - 2) + y^2 + z^2 = 4$$

(a) Find the equation of the plane tangent to the surface at the point $P = (0, 0, 2)$ on S.

(b) The equation (*) for S defines z as a function of x and y, so that $z = z(x, y)$ at least for small x, y and so that $z(0, 0) = 2$. Compute the partial derivatives $\frac{\partial z}{\partial y}(0, 0)$ and $\frac{\partial z}{\partial x}(0, 0)$.

(c) Is it possible for $z(x, y)$ to have a local minimum or local maximum at $(0, 0)$? Why or why not?

4. Let
$$F(x,y) = 3x^2ye^{x^3y}\mathbf{i} + x^3e^{x^3y}\mathbf{j}.$$

(i) Show that F is a conservative vector field in the plane (without first finding a potential).
(ii) Now find a potential of $\mathbf{F}$ (i.e. a function f so that $\nabla f = \mathbf{F}$).
(iii) Using part (ii) compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ for the curve parametrized by $\mathbf{r}(t) = t\mathbf{i} + t^{10}\mathbf{j}$, $0 \le t \le 1$.

5. Let
$$D = \{(x,y) : x^2 + y^2 \le 4\}$$
and let C be the boundary of D.
Let $\mathbf{F}(x,y) = x^3\mathbf{i} + y^3\mathbf{j}$ and let $\mathbf{n}$ be the outer normal vector.
(i) Express the integral $\int_C \mathbf{F}\cdot\mathbf{n}\, ds$ as a double integral $\iint_D g(x,y)dA$ for a suitable function g.
(ii) Find its value (it is best to use the double integral you found in (i)).
(iii) Notice that $\mathbf{F}$ is the gradient of a potential. Find this function and use it to compute $\int_C \mathbf{F}\cdot d\mathbf{r}$.

6. Let Ω be the region in 3-space given by
$$\Omega = \{(x,y,z) : a^2/4 \le x^2 + y^2 + z^2 \le a^2, x \ge 0, y \ge 0, z \ge 0, z^2/3 \le x^2 + y^2 \le z^2\}.$$

(i) Describe the set Ω in spherical polar coordinates.
(ii) Compute the integral
$$\iiint_\Omega (x^2 + y^2 + z^2)^{1/2} dV.$$

7. Let a curve be parametrized by
$$\mathbf{r}(t) = (\cos t - \sin t)\mathbf{i} + (\cos t + \sin t)\mathbf{j} + t\mathbf{k},$$
where $t \in [0, 2\pi]$.
(i) Show that the velocity vector $\mathbf{r}'(t)$ has constant magnitude. Compute the total length of the curve.
Reparametrize the curve by arclength (i.e. $\mathbf{R}(s)$ where the magnitude of $\mathbf{R}'(s)$ is 1. Specify the parameter interval for the new parameter.)
(ii) Compute the unit tangent vector $\mathbf{T}$, the principal unit normal vector $\mathbf{N}$, the binormal unit vector $\mathbf{B}$, all as functions of s.
(iii) Show that the curvature and torsion are independent of s and compute them.