1. A curve is parametrized by

$$\mathbf{r}(t) = 4\sin t\mathbf{i} + 3t\mathbf{j} + 4\cos t\mathbf{k}$$

where the ("time")-parameter is chosen from the interval $[0, 3\pi]$.

Determine (not necessarily in this order)

(i) the length of the curve,

(ii) the curvature and torsion of the curve at time t,

(iii) the unit tangent, principal unit normal and binormal vectors, as functions of t.

(iv) an equation of the form $Ax + By + Cz = D$ for the *osculating plane at time* t. This is the plane through the point on the curve parallel to both $\mathbf{T}$ and $\mathbf{N}$, at time t.

(v) For each point on the curve consider the line through this point which is parallel to the principal unit normal vector. Show these lines intersect the y axis under a constant angle (equal to $\pi/2$).

(vi) Show that all the tangent lines make a constant angle with the y axis.

(vii) Find an arclength reparametrization of the curve.

(viii) For each t find the center and radius of the *osculating circle.*

(ix) Find the *evolute* of the curve, that is the locus of the *centers of curvature* (which are the centers of the osculating circle).

2. *For extra credit:* Often a curve in the $x-y$-plane is described in polar coordinates by $\rho = \rho(\theta)$, $a \le \theta \le b$ where ρ is the distance to the origin and θ is the angle with the positive x-axis. We then have the parametrization

$$\mathbf{r}(\theta) = \rho(\theta)\cos\theta\mathbf{i} + \rho(\theta)\sin\theta\mathbf{j}.$$

a) Show that the arclength is

$$\int_a^b \sqrt{\rho^2 + (\rho')^2}d\theta$$

where the prime denotes derivative with respect to θ.

b) Show that the curvature (expressed as function of θ) is given by

$$k = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{((\rho')^2 + \rho^2)^{3/2}}.$$

3. *For extra credit.* In this problem we are given a differentiable function κ defined on an interval L. We construct a plane curve that is parametrized by arclength and the curvature for $\mathbf{r}(s)$ is given by $\kappa(s)$.

We first put $\nu(s) = \phi + \int_0^s \kappa(\sigma)d\sigma$ for a constant ϕ and then define

$$\mathbf{r}(s) = \Big(a + \int_0^s \cos(\nu(\alpha))d\alpha\Big)\mathbf{i} + \Big(b + \int_0^s \sin(\nu(\alpha))d\alpha\Big)\mathbf{j}$$

where a and b are arbitrary constants. Argue that indeed the curve parametrized by $\mathbf{r}(s)$ has curvature $\kappa(s)$ at the point $\mathbf{r}(s)$.

4. *For extra credit.* Let $\mathbf{r}$ be the parametrization of a curve with nonzero curvature but zero torsion. Show that the curve lies in a plane.