The osculating circle.

Suppose we are given a curve, parametrized by arclength

$$s \mapsto \mathbf{r}(s)$$

so that $|\mathbf{r}'(s)| = 1$ (and we assume that the derivatives up to order three are continuous).

Let P_0 be a point on the curve, corresponding to the parameter s_0, and we assume that the curvature $\kappa_0 := \kappa(s_0)$ at P_0 is not zero. Let $\mathbf{N}(s_0) = \frac{d\mathbf{T}}{ds}(s_0)$ be the principal normal vector at P_0.

Consider a number $\rho \neq 0$ and the point

$$Q(\rho, s_0) = \mathbf{r}(s_0) + \rho\mathbf{N}(s_0).$$

Let $C(\rho, s_0)$ be the circle of radius $|\rho|$ centered at

$$Q(\rho, s_0) = \mathbf{r}(s_0) + \rho\mathbf{N}(s_0).$$

This circle passes through the point P_0 and we shall see that the tangent line is exactly the same line as the tangent line to the curve at P_0.

Goal: We wish to show that if $\mathbf{r}_1(s)$ is the arclength parametrization of this circle with $\mathbf{r}_1(s_0) = \mathbf{r}(s_0)$ then $|\mathbf{r}(s) - \mathbf{r}_1(s)|$ vanishes to second order at s_0, and it vanishes to third order at s_0 *if and only if* $\rho = 1/\kappa(s_0)$. The osculating circle is then defined to be the circle centered at

$$\mathbf{r}(s_0) + \rho\mathbf{N}(s_0)$$

with radius $1/\kappa(s_0)$. The center is called the center of curvature at P_0 and the radius $1/\kappa(s_0)$ is called the radius of curvature at P_0.

We shall distinguish the cases $\rho > 0$ and $\rho < 0$. Let us first consider the (interesting) case $\rho > 0$. *Draw a picture of the circle with radius ρ, centered at $\mathbf{r}(s_0) + \rho\mathbf{N}(s_0)$:*

A parametrization (by arclength) *of this circle* is given by

$$\mathbf{r}_1(s) = \mathbf{r}(s_0) + \rho\mathbf{N}(s_0) + \rho\cos(\tfrac{s-s_0}{\rho})(-\mathbf{N}(s_0)) + \rho\sin(\tfrac{s-s_0}{\rho})(\mathbf{T}(s_0)).$$

Then

$$\mathbf{r}_1(s_0) = \mathbf{r}(s_0).$$

Also we compute that $\mathbf{r}_1'(s) = -\sin(\frac{s-s_0}{\rho})(-\mathbf{N}(s_0)) + \cos(\frac{s-s_0}{\rho})(\mathbf{T}(s_0))$ so that $\mathbf{r}_1'(s_0) = \mathbf{T}(s_0)$, hence

$$\mathbf{r}_1'(s_0) = \mathbf{r}'(s_0).$$

Thus by Taylor's formula (applied to each coordinate) and the just established facrt that $\mathbf{r}(s_0 - \mathbf{r}_1(s_0) = 0$, $\mathbf{r}'(s_0) - \mathbf{r}_1'(s_0) = 0$ we get

$$(*) \qquad \mathbf{r}(s) - \mathbf{r}_1(s) = \big(\mathbf{r}''(s_0) - \mathbf{r}_1''(s_0)\big)\frac{(s-s_0)^2}{2} + \text{Error}$$

where the error term satisfies an estimate $|\text{Error}| \le C|s-s_0|^3$ in an interval containing s_0 in its interior.

We now examine the coefficient vector of the quadratic term in the displayed formula (*). For the second derivative we get $\mathbf{r}_1''(s) = -\rho^{-1}\cos(\frac{s-s_0}{\rho})(-\mathbf{N}(s_0)) - \rho^{-1}\sin(\frac{s-s_0}{\rho})(\mathbf{T}(s_0))$ so that $\mathbf{r}_1''(s_0) = \rho^{-1}\mathbf{N}(s_0)$.

On the other hand $\mathbf{r}'' = \frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$ and evaluating this for $s = s_0$ we obtain $\mathbf{r}''(s_0) = \kappa(s_0)\mathbf{N}(s_0)$. Thus (*) can be written as

$$(**) \qquad \mathbf{r}(s) - \mathbf{r}_1(s) = (\kappa(s_0) - \rho^{-1})\mathbf{N}(s_0)\frac{(s-s_0)^2}{2} + \text{Error}$$

and the quadratic term drops out *if and only if* we choose ρ to be be equal to the reciprocal of the curvature at P_0, i.e.

$$\rho = 1/\kappa(s_0).$$

We still have to rule out the case where $\rho < 0$ and consider the circle of radius $|\rho|$ with center $\mathbf{r}(s_0) + \rho\mathbf{N}(s_0) = \mathbf{r}(s_0) - |\rho|\mathbf{N}(s_0)$ (draw a picture). To get an arclength parametrization with $\mathbf{r}(s_0) = \mathbf{r}_1(s_0)$ we have to define

$$\mathbf{r}_1(s) = \mathbf{r}(s_0) - |\rho|\mathbf{N}(s_0) + |\rho|\cos(\tfrac{s-s_0}{|\rho|})\mathbf{N}(s_0) + \rho\sin(\tfrac{s-s_0}{|\rho|})(\mathbf{T}(s_0)).$$

and we compute that indeed $\mathbf{r}(s_0) = \mathbf{r}_1(s_0)$ and $\mathbf{r}'(s_0) = \mathbf{r}_1'(s_0)$. However we get now $\mathbf{r}_1''(s_0) = -|\rho|^{-1}\mathbf{N}(s_0)$ so that $\mathbf{r}''(s_0) - \mathbf{r}_1''(s_0) = (\kappa(s_0) + |\rho|^{-1})\mathbf{N}(s_0)$ so that the quadratic term does not drop out in this case.