

### On Taylor's theorem

We first formulate Taylor's theorem on approximation by polynomials where the remainder or ("error") term is given as an integral. Other qualitative descriptions of the remainder term are given below.

**Theorem 1.** *Let  $f$  be a function having  $n+1$  continuous derivatives on an interval  $I$ . Let  $a \in I$ ,  $x \in I$ . Then*

$$(*_n) \quad f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x, a)$$

where the error is given by

$$(**_n) \quad R_n(x, a) = \frac{(x-a)^{n+1}}{n!} \int_0^1 (1-t)^n f^{(n+1)}(a+t(x-a)) dt.$$

*Proof.* For  $n=0$  this just says that

$$f(x) = f(a) + (x-a) \int_0^1 f'(a+t(x-a)) dt$$

and if we make a substitution  $s = a + t(x-a)$  (with  $ds = (x-a)dt$ ) then the formula becomes

$$f(x) = f(a) + \int_a^x f'(s) ds$$

which of course is just the fundamental theorem of calculus.

For  $n=1$  we use the formula  $(*_0)$  and integrate by parts. That is we apply the formula

$$\int_0^1 u(t)v'(t) dt = u(1)v(1) - u(0)v(0) - \int_0^1 u'(t)v(t) dt$$

with  $u(t) = f'(a+t(x-a))$ ,  $v(t) = (t-1)$ .

We then get

$$\begin{aligned} \int_0^1 f'(a+t(x-a)) dt &= f'(a+x-a)(1-1) - f'(a)(-1) - \int_0^1 \frac{d}{dt} \{f''(a+t(x-a))\} (t-1) dt \\ &= f'(a) + \int_0^1 (1-t)(x-a)f''(a+t(x-a)) dt. \end{aligned}$$

Therefore by  $(*_0)$ ,  $(**_0)$

$$\begin{aligned} f(x) &= f(a) + (x-a) \int_0^1 f'(a+t(x-a)) dt \\ &= f(a) + (x-a)f'(a) + (x-a)^2 \int_0^1 (1-t)f''(a+t(x-a)) dt. \end{aligned}$$

If  $f$  has three continuous derivatives one can iterate the procedure, integrate by parts in the last integral and get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(a) + \frac{(x-a)^3}{2} \int_0^1 (1-t)^2 f'''(a+t(x-a)) dt.$$

We prove the general case using a method called “mathematical induction”.<sup>1</sup>

We show that the formula  $(*_n)$  implies the formula  $(*)_{n+1}$  (assuming that  $f$  has  $n + 2$  continuous derivatives). Suppose we have already proved the formula for a certain number  $n \geq 0$ . Then we integrate by parts in the remainder term  $R_n(x, a)$  (cf. the above formula for integration by parts with  $u_n(t) := f^{(n+1)}(a + t(x - a))$ ,  $v_n(t) = -(1 - t)^{n+1}/(n + 1)$ ). We obtain

$$\begin{aligned} \int_0^1 (1 - t)^n f^{(n+1)}(a + t(x - a)) dt &= \int_0^1 u_n(t) v_n'(t) dt \\ &= u_n(1)v_n(1) - u_n(0)v_n(0) - \int_0^1 u_n'(t)v_n(t) dt \\ &= 0 + \frac{f^{(n+1)}(a)}{n + 1} - \int_0^1 (x - a) f^{(n+2)}(a + t(x - a)) \frac{-(1 - t)^{n+1}}{n + 1} dt \end{aligned}$$

and therefore after deviding by  $n!$  and multiplying with  $(x - a)^{n+1}$  we get

$$R_n(x, a) = \frac{(x - a)^{n+1}}{(n + 1)!} + R_{n+1}(x, a).$$

Assuming the correctness of  $(*_n)$ ,  $(**)_n$  we may thus deduce  $(*)_{n+1}$ ,  $(**_{n+1})$ :

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x - a) + \dots + \frac{f^n(a)}{n!}(x - a)^n + R_n(x, a) \\ &= f(a) + \frac{f'(a)}{1!}(x - a) + \dots + \frac{f^n(a)}{n!}(x - a)^n + \frac{f^{(n+1)}(a)}{(n + 1)!}(x - a)^{n+1} + R_{n+1}(x, a). \end{aligned}$$

with  $R_{n+1}(x, a) = \frac{(x-a)^{n+2}}{(n+1)!} \int_0^1 (1-t)^{n+1} f^{(n+2)}(a + t(x-a)) dt$ .  $\square$

### More on the remainder term

We now want to estimate the remainder term  $R_n$ .

**Theorem 2.** *Let  $f$  be as in Theorem 1 and  $R_n$  as in  $(**_n)$ . Let*

$$M(x, a) = \max\{|f^{(n+1)}(u)| : u \text{ between } a \text{ and } x\}$$

*which can also be written as  $M(x, a) = \max\{|f^{(n+1)}(a + t(x - a))| : t \text{ between } 0 \text{ and } 1\}$ .*

*Then*

$$|R_n(x, a)| \leq \frac{M(x, a)}{(n + 1)!} |x - a|^{n+1}.$$

*Proof.* We estimate

$$|R_n(x, a)| \leq \frac{|x - a|^{n+1}}{n!} \int_0^1 (1-t)^n M(x, a) dt = \frac{|x - a|^{n+1}}{n!} \frac{M(x, a)}{n + 1} = M(x, a) \frac{|x - a|^{n+1}}{(n + 1)!}$$

The last theorem can be strengthened as follows.

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<sup>1</sup>To describe this method: *One establishes that  $(*_1)$  is true and then one shows that  $(*)_n$  implies  $(*)_{n+1}$  and this implication holds for  $n = 1, 2, 3, \dots$ . As a consequence one can say that since  $(*_1)$  is true (as was shown) and since  $(*_1)$  implies  $(*_2)$  is true the statement  $(*_2)$  must be also true. But then since  $(*_2)$  is true and since  $(*_2)$  implies  $(*_3)$  is true the statement  $(*_3)$  must be also true. Then since  $(*_3)$  is true and since  $(*_3)$  implies  $(*_4)$  is true the statement  $(*_4)$  must be also true. And so on.*

**Theorem 3.** *Let  $f$  be as in Theorem 1. There is a number  $\gamma$  between  $a$  and  $x$  such that*

$$R_n(x, a) = \frac{f^{(n+1)}(\gamma)}{(n+1)!} (x-a)^{n+1}$$

*Proof.*

Let  $k$  be the minimum of  $f^{(n+1)}(a+t(x-a))$  in the interval  $[0, 1]$  and let  $K$  be the maximum of the same function in the same interval. Then

$$\int_0^1 (1-t)^n k dt \leq \int_0^1 (1-t)^n f^{(n+1)}(a+t(x-a)) dt \leq \int_0^1 (1-t)^n K dt$$

Evaluating the integral on the left and the right yields

$$\frac{k}{(n+1)} \leq \int_0^1 (1-t)^n f^{(n+1)}(a+t(x-a)) dt \leq \frac{K}{(n+1)}.$$

An application of the intermediate value theorem to the continuous function  $t \mapsto \frac{1}{n+1} f^{(n+1)}(a+t(x-a))$  yields shows that there exists a number  $\tilde{t}$  (depending on  $x$  and  $a$ ) between 0 and 1 such that

$$\frac{1}{n+1} f^{(n+1)}(a+\tilde{t}(x-a)) \int_0^1 (1-t)^n f^{(n+1)}(a+t(x-a)) dt$$

This means, by our integral formula for the remainder that

$$R_n(x, a) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(a+\tilde{t}(x-a))$$

which is what we claimed (set  $\gamma$  in the statement of the theorem to  $a+\tilde{t}(x-a)$ ).  $\square$

*Alternative expression of the remainder term:* The remainder term can also be expressed by the following formula:

$$R_n(x, a) = \frac{1}{n!} \int_a^x (x-u)^n f^{(n+1)}(u) du.$$

To see that this is equivalent with  $(**)_{n+1}$  one just makes the substitution  $u = a+s(t-a)$  (so  $du$  becomes  $(x-a)dt$  and the integral from  $a$  to  $x$  is changed to an integral over the interval  $[0, 1]$ ).

*Remark:* The conclusions in Theorem 2 and Theorem 3 are true under the assumption that the derivatives up to order  $n+1$  exist (but  $f^{(n+1)}$  is not necessarily continuous). For this version one cannot longer argue with the integral form of the remainder. You can find it in Professor Rudin's book "Principles of mathematical analysis".

**Exercises:** (i) Let  $P$  be a polynomial of degree  $N$  and let  $a$  be any real number. Then we can expand  $P$  about  $a$ :

$$P(x) = \sum_{k=0}^N \frac{P^{(k)}(a)}{k!} (x-a)^k.$$

Why is there no remainder term?

- (ii) Expand  $x^4$  in terms of powers of  $(x - \pi)$  and in terms of powers of  $(x + 2\pi)$ .  
 (iii) Show that

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + E(x)$$

where  $|E(x)| \leq x^6/720$ .

- (iv) Show that

$$\sin x = x - \frac{x^3}{6} + E(x)$$

with  $|E(x)| \leq |x|^5/120$ .

- (v) Show that for  $-1/2 < x < 1/2$

$$\sqrt{1+x} = 1 + \frac{x}{2} + E(x) \text{ where } |E(x)| \leq \frac{x^2}{2\sqrt{2}}.$$