Mathematics 522  
Fall 2013  

Homework assignment No.1.  
Due Friday, September 20. This is probably largely a review of Math 521 material.  

1. What can you say about convergence or divergence of the following series? Prove or disprove!  
   
   (i) \( \sum_{n=1}^{\infty} 3^{-n} \left( \frac{n+1}{n} \right)^2 \), \quad (ii) \( \sum_{n=2}^{\infty} \frac{1}{(\log n)^{100}} \), \quad (iii) \( \sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}} \)  
   \( (iv) \sum_{n=1}^{\infty} \frac{n^2}{3n^2} \), \quad (v) \( \sum_{n=1}^{\infty} \frac{1}{n(\log n)^p} \), \quad (vi) \( \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{\log n} + \frac{\cos n}{n^2} \right) \)  

2. For which \( x \in \mathbb{R} \) do the following series converge? (For series (i) compute the sum in case of convergence). On which sets do these series converge uniformly?  
   
   (i) \( \sum_{n=1}^{\infty} nx^{2n} \), \quad (ii) \( \sum_{n=1}^{\infty} (2^{1/n} - 1)^n x^n \), \quad (iii) \( \sum_{n=1}^{\infty} (\cos \left( \frac{2\pi}{n} \right) - 1)e^{nx} \)  

3. Consider the following sequences.  
   \( a_n = \sqrt{n^3 + n^2} - \sqrt{n^3} \)  
   \( b_n = \sqrt{n^3 + n} - \sqrt{n^3} \)  
   \( c_n = \frac{1}{n} + \frac{1}{(n+1)!} + \cdots + \frac{1}{(n^3)!} + \frac{1}{(n^3)!} \)  
   \( d_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} \)  
   \( e_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n^2} \)  
   
   For each one prove or disprove convergence of the sequence.  

4. (i) Let \( a_n \) be a sequence for which \( |a_n - a_{n+1}| \leq (n \log n)^{-1} \) for all \( n \geq 2 \). Does \( a_n \) necessarily converge?  
   (ii) Let \( b_n \) be a sequence for which \( |b_n - b_{n+1}| \leq (3/2)^{-n} \). Does \( b_n \) necessarily converge?  

5. Do the following sequences converge as \( n \to \infty \)? Prove or disprove (answers may depend on the parameters involved).  
   
   (i) \( \int_{1/n}^{1/2} \cos(x)x^a(\log \frac{1}{x})^b dx \), \quad (ii) \( \int_2^n x^a(\log x)^b dx \), \quad (iii) \( \int_0^n e^{-\sqrt{x}} dx \)  
   \( (iv) \int_3^n \frac{\cos x}{x} dx \), \quad (v) \( \int_0^n x^m \cos(x^3)dx \), \quad \( m = 0, 1, 2 \).