Mathematics 522

Homework assignment No.2.

Due Friday, September 27.

1. For complex numbers \( z \) define

\[
\sinh z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}
\]

\[
\cosh z = \sum_{\ell=0}^{\infty} \frac{z^{2\ell}}{(2\ell)!}
\]

(i) Show that these series converge for all \( z \) and thus the functions \( \sinh \) and \( \cosh \) are well defined.

(ii) what is the relation between \( \exp, \cosh, \sinh \)?

If we define \( \cos z = \cosh(iz) \), \( \sin z = \frac{1}{i} \sinh(iz) \) what is the relation between \( \exp, \cos, \sin \)?

(iii) Show that

\[
\sinh(z+w) = \sinh z \cosh w + \cosh z \sinh w
\]

\[
\cosh(z+w) = \cosh z \cosh w + \sinh z \sinh w
\]

Derive similar formulas for \( \sin(z+w), \cos(z+w) \).

2.-3.-4.-5. Problems 4,5,6 and 9 on p.165-166 in Rudin’s book.

6. Suppose the real-valued functions \( f_n \) and \( g_n \) are defined on an interval \( E \) (or more generally on a set \( E \)).

Suppose that

(a) the partial sums of the series \( \sum f_n(x) \) are uniformly bounded on \( E \),

(b) \( g_n \to 0 \) uniformly on \( E \)

(c) \( g_k(x) \geq g_{k+1}(x) \) for all \( k = 1,2,\ldots \) and all \( x \in E \).

Prove that \( \sum f_n g_n \) converges uniformly on \( E \)

7. Prove that \( \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \) converges uniformly on \([-1,1]\) and evaluate the sum. What do you get for \( x = 1 \)?

8. Evaluate \( \sum_{k=1}^{\infty} k^2 x^{2k+1} \) for \( |x| < 1 \).