Mathematics 629
Concerning outer measure on the real line
Classes 1-29, 2-1, 2-3.

Define for any subset $E \subset \mathbb{R}$

$$\mu^*(A) = \inf \sum_{j=1}^{\infty} \text{length}(I_j) \tag{1}$$

where the infimum is taken over all collections $\{I_j\}_{j=1}^{\infty}$ *of closed intervals with the property that* $A \subset \cup_{j=1}^{\infty} I_j$.

We proved that E is an outer measure in the sense of section 1.4 in Folland. The proof is a special case of Proposition 1.10 in Folland (specialize to $\mathcal{E}$ being the collection of bounded closed intervals, with the function ρ defined by $\rho(I) = \text{length}(I)$ for $I \in \mathcal{E}$).

Proposition. *For any closed interval* $[a, b]$ *we have* $\mu^*([a, b]) = b - a$.

Since we can cover $[a, b]$ with $[a, b]$ it is easy to see from the definition of μ^* that $\mu^*([a, b]) \leq b - a$. In order to finish the proof we need to show that

$$\mu^*([a, b]) \geq b - a.$$

This is harder and requires two steps.

First step. We claim the following preliminary result: Given a *finite* collection of intervals I_j, $j = 1, \dots, n$, so that $A \subset \cup_{j=1}^{n} I_j$ we have that

$$b - a \leq \sum_{j=1}^{n} \text{length}(I_j).$$

To verify this we consider the endpoints $a_1, b_1, a_2, b_2, \dots, a_N, b_N$ of these intervals and form a set Endp consisting of these endpoints. Endp is nonempty and contains at most $2n$ different points. We label them $x_0 \dots, x_N$ where $x_0 < x_1 < \cdots < x_N$ (and $N + 1 \leq 2n$). Since $[a, b] \subset \sum_{k=1}^{N} [x_{k-1}, x_k]$ we have that $x_0 \leq a$ and $b \leq x_N$.

For each $[x_{k-1}, x_k]$ there is an interval in the collection $\{I_1, \dots, I_n\}$ which contains $[x_{k-1}, x_k]$ (there may be more than one such interval). Thus for each $k \in \{1, 2, \dots, N\}$ we can pick $j(k)$ so that $[x_{k-1}, x_k] \subset I_{j(k)}$.

Notice that for fixed j the intervals $[x_{k-1}, x_k]$ with $j(k) = j$ have disjoint interior and are contained in I_j. Thus we have, for any fixed j,

$$\sum_{k:j(k)=j} (x_k - x_{k-1}) \leq \text{length}(I_j).$$

Therefore

$$b - a \leq x_N - x_0 = \sum_{k=1}^{N} (x_k - x_{k-1}) = \sum_{j} \sum_{k:j(k)=j} (x_k - x_{k-1}) \leq \sum_{j} \text{length}(I_j)$$

and we have verified the claim.

Second step: We are now given a cover of $[a,b]$ with a countable number of closed intervals $I_j = [a_j, b_j]$, $j = 1, 2, \dots$ (so that $[a,b] \subset \cup_{j=1}^{\infty} I_j$). We wish to show that

$$\sum_{j=1}^{\infty} \text{length}(I_j) \geq b - a.$$

It suffices to show that for any $\varepsilon > 0$ we have

$$\sum_{j=1}^{\infty} \text{length}(I_j) > b - a - \varepsilon.$$

We now form slightly expanded open intervals

$$\widetilde{I}_j = (a_j - \frac{\varepsilon}{10^j}, b_j + \frac{\varepsilon}{10^j})$$

and of course $[a,b]$ is contained in the union $\cup_{j=1}^{\infty} \widetilde{I}_j$ of these expanded intervals.

Now since $[a,b]$ is a *compact* set, and the $\widetilde{I}_j$ are open, there is an N so that

$$[a,b] \subset \bigcup_{j=1}^{N} \widetilde{I}_j.$$

Of course $[a,b]$ is then also contained in the union of the closures of the intervals $\widetilde{I}_j$, $j = 1, \dots, N$ (which have length $b_j - a_j + 2\varepsilon \cdot 10^{-j}$).

Thus by the first step

$$\sum_{j=1}^{N} (b_j - a_j + 2\varepsilon \cdot 10^{-j}) \geq b - a$$

which implies

$$\sum_{j=1}^{N} (b_j - a_j) \geq b - a - 2\varepsilon \sum_{j=1}^{N} 10^{-j} > b - a - \varepsilon.$$

Of course $\sum_{j=1}^{N} (b_j - a_j) \leq \sum_{j=1}^{\infty} (b_j - a_j)$ and thus also

$$\sum_{j=1}^{\infty} \text{length}(I_j) = \sum_{j=1}^{\infty} (b_j - a_j) > b - a - \varepsilon.$$

□

A discussion related to Caratheodory's theorem

Given the outer measure μ^* on a set X we define a set A to be μ^*-measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \text{for all } E \subset X.$$

One also says that A is "Caratheodory measurable" if μ^* is given.

Caratheodory's **Theorem 1.11** states that the μ^*-measurable sets form a σ-algebra $\mathcal{M}$, and the restriction of μ^* to $\mathcal{M}$ is a measure on $\mathcal{M}$. We will cover this in class.

It is not immediate to see how large and therefore how useful this σ-algebra is. Let us therefore consider the *special case* of the outer measure on the real line defined above in equation (1) and prove

Proposition. *If μ^* is as in (1) then the σ-algebra $\mathcal{M}$ of μ^*-measurable sets contains the Borel algebra.*

Proof. The Borel algebra is generated by the rays (c, ∞), $c \in \mathbb{R}$, see Proposition 1.2 in Folland. We therefore have to just prove that each ray is μ^*-measurable (because $\mathcal{M}$ includes all rays and by Lemma 1.1 all Borel sets).

Let $A = (c, \infty)$. By the monotonicity property of the outer measure we have $\mu^*(E) \le \mu(E \cap A) + \mu(E \cap A^c)$. We have to show the opposite inequality $\mu^*(E) \ge \mu(E \cap A) + \mu(E \cap A^c)$ which follows from

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \le \mu^*(E) + \varepsilon \tag{2}$$

for arbitrary $\varepsilon > 0$.

Let I_j be a collection of closed intervals with $\sum_{j=1}^{\infty} \text{length}(I_j) < \mu^*(E) + \varepsilon/2$. Let $I_j^{\text{left}} = I_j \cap (-\infty, c]$ and let $I_j^{\text{right}} = I_j \cap [c - \varepsilon 10^{-j}, \infty)$. Then I_j^{left} and I_j^{right} are closed intervals with $I_j \subset I_j^{\text{left}} \cup I_j^{\text{right}}$ and

$$\text{length}(I_j^{\text{left}}) + \text{length}(I_j^{\text{right}}) \le \text{length}(I_j) + \varepsilon 10^{-j}.$$

Hence

$$\sum_{j=1}^{\infty} \text{length}(I_j^{\text{left}}) + \sum_{j=1}^{\infty} \text{length}(I_j^{\text{right}}) \le \Big(\sum_{j=1}^{\infty} \text{length}(I_j)\Big) + \varepsilon/9.$$

Now $E \cap A \subset \cup_{j=1}^{\infty} I_j^{\text{right}}$ and $E \cap A^c \subset \cup_{j=1}^{\infty} I_j^{\text{left}}$. By definition of μ^* we have $\mu^*(E \cap A) \le \sum_{j=1}^{\infty} \text{length}(I_j^{\text{right}})$ and $\mu^*(E \cap A^c) \le \sum \text{length}(I_j^{\text{left}})$. Hence

$$\begin{aligned}
\mu^*(E \cap A) + \mu^*(E \cap A^c) &\le \sum_{j=1}^{\infty} \text{length}(I_j^{\text{left}}) + \sum_{j=1}^{\infty} \text{length}(I_j^{\text{right}}) \\
&\le \Big(\sum_{j=1}^{\infty} \text{length}(I_j)\Big) + \varepsilon/9 \\
&\le \mu^*(E) + \varepsilon/2 + \varepsilon/9
\end{aligned}$$

and thus (2) is verified.

Comment. The outer measure defined above is translation invariant. That is, if E is given and for any x_0 we define the x_0-translate $x_0 + E$ as the set of points x which can be written as $x = x_0 + y$ with $y \in E$ then $\mu^*(E) = \mu^*(x_0 + E)$. Prove this.

A combination of the two propositions in this handout (together with Caratheodory's theorem 1.11) shows that μ^* restricted to the Borel σ-algebra $\mathcal{B} \equiv \mathcal{B}(\mathbb{R})$ defines a translation invariant measure on $\mathcal{B}$ which satisfies $\mu([a, b]) = b - a$.

Moreover, since the σ-algebra of μ^*-measurable functions is complete it contains the completion of the Borel σ-algebra (i.e. the Lebesgue measurable sets on the real line). μ^* restricted to the σ-algebra of Lebesgue measurable sets is named "Lebesgue-measure".

A homework problem.

Approximation of real numbers with rational numbers.

In the following problem, part (i)* is considered background or motivation. It is not related to Math 629 and can be solved by a suitable application of the pidgeonhole principle.

Part (ii) is related to Math 629 and assigned as homework.

(i)* One can show that for every $x \in \mathbb{R}$ there are infinitely many fractions $\frac{p}{q}$ so that

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{q^2}.$$

(ii) Concerning the sharpness of (i)*:

Let $\epsilon > 0$ and let A_ε be the set of all real numbers which have the property that there are infinitely many fractions $\frac{p}{q}$ so that $\left|x - \frac{p}{q}\right| \leq q^{-2-\epsilon}$. Then the set A_ε has Lebesgue measure zero.

Hint: Consider first $A_\varepsilon \cap [0, 1]$ and use the Borel-Cantelli Lemma (which says that if E_k are measurable sets with $\sum_k \mu(E_k) < \infty$ then $\mu(\limsup E_k) = 0$.)