Mathematics 629
Review problems

1. Let f be a measurable real-valued function on a measure space $(X, \mathcal{M}, \mu)$ so that f is finite almost everywhere. Prove:

For every set $E \subset X$ of finite measure we have

$$\lim_{\alpha \to \infty} \mu(\{x \in E : |f(x)| > \alpha\}) = 0.$$

2. (i) (i) Review the statement and proof of the Borel-Cantelli lemma.

(ii) Let f_n be a sequence of measurable functions on a finite measure space X, so that $|f_n(x)| < \infty$ for almost every $x \in X$.

Show that there is a sequence c_n of positive real numbers so that

$$\lim_{n \to \infty} \frac{f_n(x)}{c_n} = 0 \text{ almost everywhere.}$$

Hint: Show that you can pick c_n so that $\mu(\{x : |f_n(x)/c_n| > 1/n\}) < 2^{-n}$. Then apply the Borel-Cantelli Lemma.

3. Let f be a continuous function on the interval $[0, 1]$. Let

$$\Gamma = \{(x, y) : y = f(x), \quad 0 \le x \le 1.\}$$

(i) Denote by m the Lebesgue measure in $\mathbb{R}^2$ and prove that $m(\Gamma) = 0$.

(ii) What can you say if f is merely assumed to be Riemann integrable on $[0, 1]$?

4. If f is integrable on $\mathbb{R}$ define

$$F(x) = \int_{-\infty}^{x} f(s) ds$$

and show that F is *uniformly continuous*.

5. Integrability of f on $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$.

(i) Show that there is a continuous function f on $\mathbb{R}$ so that f is integrable but yet $\limsup_{x \to \infty} f(x) = \infty$.

(ii) If f is uniformly continuous on $\mathbb{R}$ and integrable on $\mathbb{R}$ then $\lim_{|x| \to \infty} f(x) = 0$.

6. (i) Suppose f is integrable on $\mathbb{R}$, real valued and $\int_E f(x)dx \ge 0$ for every measurable set E. Prove that $f(x) \ge 0$ almost everywhere.

(ii) Suppose f is integrable on $\mathbb{R}$, real valued and $\int_E f(x)dx = 0$ for every measurable set E. Prove that $f(x) = 0$ almost everywhere.

7. Prove: If $f \in L^1(\mathbb{R})$ and $\epsilon > 0$ then there is a simple function s so that $\int |f(x) - s(x)| \, dm < \epsilon$.

8. Suppose f is an integrable function on $\mathbb{R}$. Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\cos(nx)\,dx = 0\,.$$

Hint: Prove this first for characteristic functions of intervals. Then for characteristic functions of open sets with finite measure, then for simple functions, and then for general $f \in L^1$.

9. For real numbers a and b and all $x \in \mathbb{R}^n$ define

$$f_{a,b}(x) = \frac{|x|^a}{1+|x|^b}.$$

For which a, b is $f_{a,b}$ integrable in $\mathbb{R}^n$?

10. Which of the following functions is integrable (the answer may depend on the given real parameters a, p, b).
(i) $f(x) = x^a$ on $[0,\infty]$.
(ii) $f(x) = x^{-1}(\ln x)^{-b}$ on $[2,\infty]$.
(iii) $f(x) = x^{-1}(\ln x)^{-b}$ on $[0,\frac{1}{2}]$.
(iv) $f(x) = x^{-1}(\ln x)^{-b}$ on $[\frac{1}{2},2]$.
(v) $f(x) = |x|^p\cos(|x|^2)e^{-|x|}$ on $\mathbb{R}^n$.
(vi) $f(x) = \frac{\sin x}{x}$ on $\mathbb{R}$.
(vii) $f(x) = |x|^p\sin(x^2)e^{-x^2}$ on $\mathbb{R}$.