The Uniform boundedness principle

The following theorem is known as the Theorem of Banach and Steinhaus who published a proof based on the Baire category theorem.

**Theorem.** Let $B$ be a Banach space and $Y$ be a normed space and $G$ be a family of bounded operators from $B$ to $Y$. Suppose that for every $f \in B$ there is a constant $C(f) < \infty$ so that

$$\sup_{T \in G} \|Tf\|_Y \leq C(f).$$

Then one also has

$$\sup_{T \in G} \|T\| < \infty.$$

Here $\|T\| \equiv \|T\|_{B \to Y} := \sup\{\|Th\|_Y : \|h\|_B = 1\}$.

Textbooks nowadays base the proof on the Baire category theorem. However the original proofs were essentially constructive. The proof below is due to Hahn\(^1\), who gives credit for the idea to Lebesgue. Hahn assumed that the operators $T_n$ are linear functionals (i.e. $Y = \mathbb{R}$ or $Y = \mathbb{C}$); this was later generalized. The method of proof is known as the **gliding hump method**.

**Proof.** We argue by contradiction and assume that $\sup_{T \in G} \|T\| = \infty$. The goal now is to construct an example $f \in B$ for which $\sup_{T \in G} \|Tf\|_Y = \infty$, which achieves a contradiction.

We pick $T_1 \in G$ with $\|T_1\| \geq 1$ and $f_1 \in B$ so that

$$\|T_1f_1\|_Y \geq 1, \quad \|f_1\|_B \leq 1.$$

We construct sequences of operators $\{T_n\}_{n \in \mathbb{N}}$ with $T_n \in G$ and vectors $f_n \in B$ so that for $n \geq 2$

$$\|T_nf_n\|_Y \geq n + \sum_{k=1}^{n-1} C(f_k)$$

$$\|f_n\|_B \leq 2^{-n} \min_{1 \leq j \leq n-1} \|T_j\|^{-1}$$

Indeed suppose that $n \geq 2$ and the $T_1, \ldots, T_{n-1}$, $f_1, \ldots, f_{n-1}$ are already chosen. Since we assume $\sup_{T \in G} \|T\|_{B \to Y} = \infty$ we can pick an

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\(^1\)Hans Hahn, Über Folgen linearer Operationen. Monatsh. Math. Phys. 32 (1922), no. 1
operator in $\mathcal{G}$, labeled $T_n$, and $h_n \in B$ with $\|h_n\|_B = 1$ so that
\[
\|T_n h_n\|_Y \geq 2^{n+1} \left( \frac{n + \sum_{k=1}^{n-1} C(f_k)}{\min_{1 \leq j \leq n-1} \|T_j\|^{-1}} \right).
\]
The above statement then follows with $f_n = h_n 2^{-n} \min_{1 \leq j \leq n-1} \|T_j\|^{-1}$.
Since by the induction hypothesis $\|T_j\| \geq 1$ for $j \leq n - 1$ we obviously get $\|T_n\| \geq 2^{n+1} n \gg 1$.

By construction we have $\|f_n\| \leq 2^{-n}$ and by the completeness of $B$ we see that $f := \lim_{N \to \infty} \sum_{n=1}^{N} f_n$ exists in $B$.

We now check that
\[
\|T_n f\|_Y \geq n - 1
\]
which is a contradiction to the assumption $\sup_n \|T_n f\| \leq \sup_{T \in \mathcal{G}} \|T f\|_Y \leq C(f) < \infty$.

$T_n$ is a bounded operator and therefore $T_n f = \sum_{k=1}^{\infty} T_n f_k$ with convergence in $Y$. To prove a lower bound we show that in $\sum_{k=1}^{\infty} T_n f_k$ the main contribution comes from the term $T_n f_n$ (the “hump”) and we estimate, by the triangle inequality,
\[
\|T_n f\|_Y \geq \|T_n f_n\|_Y - \sum_{n-1}^{\infty} \|T_n f_k\|_Y - \sum_{k=n+1}^{\infty} \|T_n f_k\|_Y.
\]

For the last term we use the upper bounds for the functions $f_k$, namely
\[
\sum_{k=n+1}^{\infty} \|T_n f_k\|_Y \leq \|T_n\| \sum_{k=n+1}^{\infty} \|f_k\|_B \leq \|T_n\| \sum_{k=n+1}^{\infty} 2^{-k} \min_{1 \leq j \leq k-1} \|T_j\|^{-1} \|T_n\| \leq \|T_n\| \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n} < 1.
\]

By construction of $T_n$
\[
\|T_n f_n\|_Y - \sum_{k=1}^{n-1} \|T_n f_k\|_Y \geq n + \sum_{k=1}^{n-1} [C(f_k) - \|T_n f_k\|_Y] \geq n
\]
(since $C(f_k) = \sup_{T \in \mathcal{G}} \|T f_k\|_Y \geq \|T_n f_k\|_Y$). By the two previous displays,
\[
\|T_n f\|_Y \geq \|T_n f_n\|_Y - \sum_{j=1}^{n-1} \|T_n f_j\|_Y - \sum_{k=n+1}^{\infty} \|T_n f_k\|_Y \geq n - 1.
\]
Example: Limiting methods

We are given an infinite matrix $A = (a_{nk})_{n,k=1,2,...}$ with the property that $\sum_{k=1}^{\infty} a_{nk} x_k$ converges whenever $\lim_{k \to \infty} x_k$ exists (this is certainly true if $\sum_{k=1}^{\infty} |a_{nk}| < \infty$ for all $n \in \mathbb{N}$).

By assumption we may attach to every convergent sequence $x_n$ a new sequence $y_n = \{y_n\}$ with $y = Ax$, i.e. $y_n = \sum_{k=1}^{\infty} a_{nk} x_k$ for $n \in \mathbb{N}$.

**Definition.** The matrix $A$ defines a limiting method (also known as a "summability method") if the sequence $y_n = \{y_n\}$ for $y = Ax$ has limit $L$ whenever the sequence $x_n$ has limit $L$. In other words $\lim_{n \to \infty} (Ax)_n = \lim_{n \to \infty} x_n$ whenever the latter limit exists.

**Question:** Can one characterize those matrices $A$ for which $A$ defines a limiting method?

Indeed one can; there are two obvious necessary conditions and another not so obvious necessary condition, and these three conditions will be also sufficient. The necessary conditions are listed in the following definition.

**Definition.** The infinite matrix $A$ is called regular if the following conditions are satisfied.

(i) $\lim_{n \to \infty} a_{nk} = 0$, for every $k \in \mathbb{N}$.

(ii) $\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} = 1$

(iii) $\sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| < \infty$.

The uniform condition (iii) is stronger than the finiteness condition on $\sum_{k=1}^{\infty} |a_{nk}|$ mentioned in parentheses above.

**Example:** The Cesaro means $y_n = \frac{x_1 + \cdots + x_n}{n}$ define a regular method; here $a_{nk} = 1/n$ if $1 \leq k \leq n$ and $a_{nk} = 0$ for $k > n$.

**Toeplitz’ theorem.** A matrix $A$ defines a limiting method if and only if it is regular.

**Proof:** The interesting part of the proof is the necessity. We thus assume that $A$ defines a limiting method. Condition (i) seen to be necessary by testing against the sequence $\delta^k$, with $\delta^k_k = 1$ and $\delta^k_n = 0$ if $n \neq 0$. Condition (ii) seen to be necessary by testing against the sequence $(1, 1, 1, ...)$.

\[\text{2this terminology makes sense if the } x_n \text{ are the partial sums of a series}\]
To prove the necessity of (iii) we first prove that $\sum_{k=1}^{\infty} |a_{nk}| < \infty$ for all $n \in \mathbb{N}$.

Consider the normed space $\ell_\infty$ of convergent sequences as a closed (and thus complete) subspace of the complete space $\ell^\infty$ (with the sup-norm). For each $n \in \mathbb{N}$ and each $M \in \mathbb{N}$ define a linear functional $\lambda_{n,M}$ on $\ell_\infty$ by

$$\lambda_{n,M}(x) = \sum_{k=1}^{M} a_{nk}x_k.$$ 

Clearly $|\lambda_{n,M}(x)| \leq \sum_{j=1}^{M} |a_{nj}||x||_\infty$. The constant $\sum_{j=1}^{M} |a_{nj}|$ is sharp as one can test the inequality of the sequence with $x_k = \text{sign}(a_{nk})$ for $k \leq M$ and $x_k = 0$ for $k > M$. Thus

$$\|\lambda_{n,M}\| = \sum_{j=1}^{M} |a_{nj}|.$$ 

Now fix $x \in \ell_\infty$. Then by assumption on $A$ the sum $\sum_{k=1}^{\infty} a_{nk}x_k$ converges for every $n$. Thus for fixed $n$ we have

$$\sup_M |\lambda_{n,M}(x)| \leq C(n, x) < \infty.$$ 

By the uniform boundedness principle we also get $\sup_M \|\lambda_{n,M}\| \leq C(n) < \infty$, i.e. $\sum_{k} |a_{nk}| < \infty$ for every $n \in \mathbb{N}$.

This means that for every $n \in \mathbb{N}$

$$\lambda_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k$$

defines a bounded linear functional on $\ell_\infty$ (indeed on $\ell^\infty$); moreover by testing as above we see that

$$\|\lambda_n\| = \sup_M \sum_{k=1}^{M} |a_{nk}| = \sum_{k=1}^{\infty} |a_{nk}|.$$ 

Now if $A$ is a limiting method then the $\lim_{n \to \infty} \lambda_n(x)$ exists for every $x \in \ell_\infty$ and this limit is equal to $\lim_{n \to \infty} x_n$. In particular we have

$$\sup_{n \in \mathbb{N}} |\lambda_n(x)| \leq C(x) < \infty$$

for every $x \in \ell_\infty$. By the uniform boundedness principle we have $\sup_n \|\lambda_n\| < \infty$ and hence $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ which is the necessary condition (iii).

Proof of sufficiency. This is a standard argument in calculus, similar to the one for the example of Cesaro means, and also similar to the argument used for approximations of the identity.
We assume that the limiting method is regular and we have to show that for any \( x \in \mathbb{C} \) with \( \lim_{n \to \infty} x_n = L \) we also have \( \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} x_k = L \).

Thus let \( \varepsilon > 0 \) and we are going to verify that \( \left| \sum_{k=1}^{\infty} a_{nk} x_k - L \right| < \varepsilon \) for large \( n \).

Let \( M = \sup_n \sum_{k=1}^{\infty} |a_{nk}| \). We split
\[
\sum_{k=1}^{\infty} a_{nk} x_k - L = \sum_{k=1}^{\infty} a_{nk} (x_k - L) + L(-1 + \sum_{k} a_{nk}).
\]

By condition (ii) there is \( N_1(\varepsilon) \) so that \( |L(-1 + \sum_{k} a_{nk})| < \varepsilon/3 \) for \( n \geq N_1(\varepsilon) \).

Since \( \lim x_n = L \), there is \( N_2(\varepsilon) \) so that \( |x_k - L| < \varepsilon/(3M) \) for \( k \geq N_2(\varepsilon) \). We then have for all \( n \in \mathbb{N} \)
\[
\left| \sum_{k \geq N_2(\varepsilon)} a_{nk} (x_k - L) \right| \leq \sum_{k \geq N_2(\varepsilon)} |a_{nk}| |(x_k - L)| \leq \frac{\varepsilon}{3M} \sum_{k=1}^{\infty} |a_{nk}| \leq \frac{\varepsilon}{3}.
\]

We still have to take care of the terms \( \sum_{1 \leq k \leq N_2(\varepsilon)} a_{nk} (x_k - L) \). But for each \( k \) we have \( a_{nk} \to 0 \) as \( n \to \infty \) by condition (i). Thus also \( \lim_{n \to \infty} \sum_{1 \leq k \leq N_2(\varepsilon)} a_{nk} (x_k - L) = 0 \) and there is \( N_3(\varepsilon) \) so that \( \left| \sum_{1 \leq k \leq N_3(\varepsilon)} a_{nk} (x_k - L) \right| < \varepsilon/3 \) for \( n \geq N_3(\varepsilon) \). We can combine the three \( \varepsilon/3 \)-estimates provided that \( n \geq \max_{i=1,2,3} N_i(\varepsilon) \). \( \square \)
Example: Divergence of Fourier series.

We consider the partial sums of the Fourier series for a continuous 1-periodic function, given by

$$S_n f(a) = \sum_{k=-n}^{n} \hat{f}_k e^{2\pi i ka} \quad \text{where} \quad \hat{f}_k = \int_{0}^{1} f(t) e^{-2\pi i kt} dt.$$ 

Fix \(a\). We show that there is a 1-periodic continuous function for which the sequence \(S_n f(a)\) does not converge and is in fact unbounded. Replacing \(f\) with \(f(\cdot - a)\) we may assume \(a = 0\).

Let

$$D_n(x) = \sum_{k=-n}^{n} e^{2\pi i kx} = 1 + 2 \sum_{l=1}^{n} \cos(2\pi lx),$$

the Dirichlet kernel. After summing geometric series (or, alternatively, applying trigonometric identities) one computes

$$D_n(x) = \frac{\sin((2n + 1)\pi x)}{\sin(\pi x)}.$$ 

We consider the linear functionals \(\lambda_n\) acting on the space of 1-periodic continuous functions (with the sup-norm) by

$$\lambda_n(f) = S_n f(0) = \int_{0}^{1} D_n(t) f(t) dt.$$ 

Clearly \(|S_n f(0)| \leq \int_{0}^{1} |D_n(t)| dt \|f\|_\infty\) so that \(\|\lambda_n\| \leq \int_{0}^{1} |D_n(t)| dt\). In fact, one can check that equality holds in the last estimate. Note that for \(g_n(t) = \text{sign}(D_n(t))\) equality is achieved but \(g_n\) is not continuous. However it is straightforward to construct a sequence of 1-periodic continuous functions \(g_{n,j}\) so that \(|g_{n,j}(x)| \leq 1\) for all \(x\), \(\|g_{n,j}\|_\infty = 1\) and \(g_{n,j}\) converges to \(g_n\) almost everywhere. Thus

$$\int_{0}^{1} D_n(t) g_{n,j}(t) dt \rightarrow \int_{0}^{1} D_n(t) g_n(t) dt$$

and we conclude

$$\|\lambda_n\| = \int_{0}^{1} |D_n(t)| dt.$$ 

A straightforward calculation shows that \(\int_{0}^{1} |D_n(t)| dt \geq c \log n\) so that \(\{\|\lambda_n\|\}\) is unbounded. By the uniform boundedness principle there exists (and by examining and slightly modifying Hahn’s proof one can indeed construct) a continuous 1-periodic function \(f\) for which \(\sup_{n \in \mathbb{N}} |S_n f(0)| = \infty\).

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\(^3\)Such an example was first given in 1876 by du Bois Reymond.