

# ON SPACE-TIME ESTIMATES FOR THE SCHRÖDINGER OPERATOR

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ABSTRACT. We prove mixed norm space-time estimates for solutions of the Schrödinger equation, with initial data in  $L^p$  Sobolev (or Besov) spaces, and clarify the relation with adjoint restriction.

## 1. INTRODUCTION

We consider the Schrödinger equation,  $i\partial_t u + \Delta u = 0$ , with initial data  $u(\cdot, 0) = f$ . When  $f$  is a Schwartz function, the solution can be written as  $u = Uf$ , where

$$(1.1) \quad Uf(x, t) \equiv e^{it\Delta} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{-it|\xi|^2 + i\langle x, \xi \rangle} d\xi$$

and  $\widehat{\cdot}$  denotes the Fourier transform defined by  $\widehat{f}(\xi) = \int f(y) e^{-i\langle y, \xi \rangle} dy$ . We fix a compact time interval  $I$  and let  $L^q(\mathbb{R}^d; L^r(I))$  be the space equipped with the mixed norm

$$\|u\|_{L^q(\mathbb{R}^d; L^r(I))} = \left( \int_{\mathbb{R}^d} \left( \int_I |u(x, t)|^r dt \right)^{q/r} dx \right)^{1/q}.$$

Our aim is to bound the solution in this space when the initial data are given in Sobolev spaces  $L^p_\alpha$ , with norm  $\|f\|_{L^p_\alpha} = \|(I - \Delta)^{\alpha/2} f\|_{L^p(\mathbb{R}^d)}$ . We shall always assume that  $q, r \geq 2$ , and we will mostly assume  $p \geq 2$  as well. The cases  $r = 2$ ,  $r = q$  and  $r = \infty$  are of particular interest.

First we state some necessary conditions.

**Proposition 1.1.** *Let  $2 \leq p, q, r \leq \infty$ , and suppose that there constant  $C$  such that*

$$(1.2) \quad \|Uf\|_{L^q(\mathbb{R}^d; L^r(I))} \leq C \|f\|_{L^p_\alpha(\mathbb{R}^d)}$$

whenever  $f \in L^p_\alpha(\mathbb{R}^d)$ . Then

- (i)  $p \leq q$ ,
- (ii)  $\alpha \geq d(1 - \frac{1}{p} - \frac{1}{q}) - \frac{2}{r}$ ,
- (iii)  $\alpha \geq \frac{1}{q} - \frac{1}{r}$ ,
- (iv)  $\alpha \geq \frac{1}{q} - \frac{1}{p}$ ,
- (v)  $\alpha > \frac{1}{q} - \frac{1}{p}$  if  $r > 2$ ,
- (vi)  $\alpha > 0$  if  $r = 2$ ,  $p = q > 2$ ,  $d \geq 2$ .

The proposition can be strengthened by replacing the Sobolev norm by the Besov norm  $B^p_{\alpha, \nu}$ , for any  $\nu > 0$ , where  $\|f\|_{B^p_{\alpha, \nu}} = (\sum_{k \geq 0} 2^{k\alpha\nu} \|P_k f\|_p^\nu)^{1/\nu}$ . Here, for  $k \geq 1$ , the operators  $P_k$  localize frequencies to annuli of width  $\approx 2^k$  and  $P_0 = I - \sum_{k \geq 1} P_k$ . Recall that  $B^p_{\alpha, \nu}$  is contained in  $L^p_\alpha$  for  $\nu \leq \min\{2, p\}$ .

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*Remarks.* The inequality (1.2) has been considered in many special cases and some of the necessary conditions in Proposition 1.1 are related to similar conditions for other problems in harmonic analysis. In what follows we set  $\alpha_{cr}(p; q, r) := d(1 - \frac{1}{p} - \frac{1}{q}) - \frac{2}{r}$ .

(a) If  $p = 2$ , then the condition (ii) coincides with (iii) if  $\frac{d+1}{q} + \frac{1}{r} = \frac{d}{2}$ . This is the condition in the endpoint version of Planchon's conjecture (cf. [21], [17]).

(b) If  $p = 2$  and  $r = \infty$ , then the condition (iii) follows from the necessary conditions for Carleson's problem [5, 26], via an equivalence between local and global estimates [22].

(c) If  $p = q$  and  $2 \leq r \leq q$ , then the condition  $\alpha \geq \alpha_{cr}(p; p, r)$  is more restrictive than (vi) if  $d(\frac{1}{2} - \frac{1}{p}) - \frac{1}{r} > 0$ . In particular, if  $r = 2$ , and  $\alpha = \alpha_{cr}(p; p, 2)$ , the range  $p > \frac{2d}{d-1}$  is necessary (in analogy to the Bochner–Riesz conjecture in  $\mathbb{R}^d$ ), and for  $r = p$ ,  $\alpha = \alpha_{cr}(p; p, p)$  the range  $p > \frac{2(d+1)}{d}$  is necessary (as in the equivalent adjoint restriction problem for the paraboloid in  $\mathbb{R}^{d+1}$ , cf. Theorem 1.6 below).

(d) If  $p < q$ ,  $r = 2$ , the condition  $\alpha \geq \alpha_{cr}(p; q, 2)$  is more restrictive than (iv) if  $\frac{d+1}{q} \leq \frac{d-1}{p'}$ , the familiar range for the adjoint restriction theorem for the sphere in  $\mathbb{R}^d$ . Likewise if,  $p < q = r$  then the condition  $\alpha \geq \max\{0, \alpha_{cr}(p; q, q)\}$  implies  $\frac{d+2}{q} \leq \frac{d}{p'}$ , the range for the adjoint restriction theorem for the paraboloid in  $\mathbb{R}^{d+1}$ .

(e) The necessity of the strict inequalities in (v), (vi) of Proposition 1.1 is proved by considerations which involve the Besicovich set (see §3). The necessity of the condition (vi) in dimensions  $d \geq 2$  comes from the fact that a sharp square function estimate for the Schrödinger operator implies sharp bounds on Bochner–Riesz multipliers. The necessity for the open range (v) in one dimension was left open in [16].

**Theorem 1.2.** *Let  $2 \leq p < \infty$ .*

*Then  $U : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R}; L^r(I))$  is bounded if and only if  $r \leq 2$ .*

The sufficiency of the condition follows from [16, Proposition 5.2]. The necessity is a consequence of the following more precise bounds for frequency localized functions which also illustrates the sharpness of the necessary conditions of Proposition 1.1 (at least in the cases  $r \leq q$  and  $d = 1$ ).

**Theorem 1.3.** *For large  $\lambda$ , let*

$$\mathfrak{A}_\lambda(p; q, r) = \sup \left\{ \|Uf\|_{L^q(\mathbb{R}; L^r(I))} : \|f\|_p \leq 1, \quad \text{supp } \widehat{f} \subset \{\xi : \lambda/5 \leq |\xi| \leq 15\lambda\} \right\}.$$

*Then for  $\lambda \gg 1$ , the following norm equivalences hold.*

(i) *For  $2 \leq r \leq p \leq q \leq \infty$ ,*

$$\mathfrak{A}_\lambda(p; q, r) \approx \begin{cases} \lambda^{1/q-1/p} [\log \lambda]^{1/2-1/r} & \text{if } \frac{1}{q} + \frac{1}{r} \geq \frac{1}{2}, \\ \lambda^{1-1/p-1/q-2/r} & \text{if } \frac{1}{q} + \frac{1}{r} < \frac{1}{2}. \end{cases}$$

(ii) *For  $2 \leq p < r \leq q \leq \infty$ ,*

$$\mathfrak{A}_\lambda(p; q, r) \approx \begin{cases} \lambda^{1/q-1/r} & \text{if } \frac{2}{q} + \frac{1}{r} \geq 1 - \frac{1}{p}, \\ \lambda^{1-1/p-1/q-2/r} & \text{if } \frac{2}{q} + \frac{1}{r} < 1 - \frac{1}{p}. \end{cases}$$

Using a result in [16], (stated as Proposition 5.1 below), one can obtain sharp estimates for functions in Sobolev and Besov spaces. In order to compare such results recall that  $B_{\alpha, q_1}^p \subset B_{\alpha, q_2}^p$  for  $q_1 < q_2$ , that  $B_{\alpha, 2}^p \subset L_\alpha^p \subset B_{\alpha, p}^p$  when  $p \geq 2$ , and that  $B_{\alpha, p}^p$  is the same as the Sobolev–Slobodecki space  $W^{\alpha, p}$  when  $0 < \alpha < 1$ .

**Corollary 1.4.** *Suppose that  $2 \leq r \leq p \leq q$ ,  $\frac{1}{q} + \frac{1}{r} < \frac{1}{2}$ , or  $2 \leq p < r \leq q$ ,  $\frac{2}{q} + \frac{1}{r} < 1 - \frac{1}{p}$ . Then  $U : B_{\alpha,q}^p(\mathbb{R}) \rightarrow L^q(\mathbb{R}; L^r(I))$  is bounded with  $\alpha = 1 - \frac{1}{p} - \frac{1}{q} - \frac{2}{r}$ .*

When  $p = q$  one could hope for the following estimates.

**Conjecture 1.5.** *Let  $p \in [2, \infty)$ ,  $r \in [2, \infty]$  satisfy  $\frac{d}{p} + \frac{1}{r} < \frac{d}{2}$  and  $\frac{2d+1}{p} + \frac{1}{r} < d$ . Then  $U : B_{\alpha,p}^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d; L^r(I))$  is bounded with  $\alpha = d(1 - \frac{2}{p}) - \frac{2}{r}$ .*

To prove the conjecture it would suffice to prove the sharp estimates with  $r = \infty$ ,  $p$  and 2. The estimates with  $r = \infty$  strengthen the sharp  $L^p$ -Sobolev bounds for fixed  $t$  and  $\alpha = 2d|1/2 - 1/p|$  due to Fefferman–Stein [10] and Miyachi [18]. In [16], the conjecture was proven in the reduced range  $p \in (\frac{2(d+2)}{d}, \infty)$ , and for  $d = 1$  it was proven in the range  $p \in (4, \infty)$ . In [24], the conjecture was proven for  $p \in (\frac{2(d+3)}{d+1}, \infty)$  with  $r \geq p$ ; moreover a related result was proven for the semigroups  $\exp(it(-\Delta)^{a/2})$  for  $a > 1$ . A nonendpoint result for  $a = 2$ ,  $p = r$  had been previously obtained in [22].

In the case of the Schrödinger semigroup ( $a = 2$ ) it is well known that the local smoothing and maximal inequalities are closely related to estimates for the adjoint restriction operator for a compact portion of the paraboloid in  $\mathbb{R}^{d+1}$  (see [25], [30], [3], [15], [22]). Here we improve the known  $L^q(L^r)$  bounds for  $q = r$  by establishing an actual equivalence of the space-time regularity estimates with estimates for the adjoint restriction operator (a related result establishing the equivalence between adjoint restriction and Bochner-Riesz for paraboloids was found by Carbery [4]).

Let  $\mathcal{E}$  denote the adjoint restriction (or Fourier extension) operator given by

$$(1.3) \quad \mathcal{E}f(\xi, s) = \int_{|y| \leq 1} f(y) e^{is|y|^2 - i\langle \xi, y \rangle} dy, \quad (\xi, s) \in \mathbb{R}^d \times \mathbb{R}.$$

**Definition.** *We say that  $R^*(p \rightarrow q)$  holds true if  $\mathcal{E} : L^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^{d+1})$  is bounded.*

In the critical case  $q(p) = \frac{d+2}{d}p'$  it follows from the explicit formula

$$(1.4) \quad Uf(x, t) = \frac{1}{(4\pi it)^{d/2}} \int \exp\left(\frac{i|x-y|^2}{4t}\right) f(y) dy$$

and scaling that  $R^*(p \rightarrow q(p))$  implies the  $L^p(\mathbb{R}^d) \rightarrow L^{q(p)}(\mathbb{R}^d \times I)$  boundedness of  $U$ . Moreover it was also shown in [22] that it implies the  $L_\alpha^p \rightarrow L^p(\mathbb{R}^d \times I)$  bound for  $\alpha > 2d(\frac{1}{2} - \frac{1}{p}) - \frac{2}{p}$ . We strengthen these results as follows.

**Theorem 1.6.** *Suppose  $2 \leq p \leq q < \infty$ . Then the following are equivalent:*

- (i)  $R^*(p \rightarrow q)$  holds.
- (ii) *The operator  $U : B_{\alpha,1}^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d \times I)$  is bounded with  $\alpha = d(1 - \frac{1}{p} - \frac{1}{q}) - \frac{2}{q}$ .*

Again, using Proposition 5.1, we can also obtain results on larger spaces (including the Sobolev space  $L_\alpha^p$ ) if we give up an endpoint in the  $q$ -range.

**Corollary 1.7.** *Let  $2 < q_0 < \infty$ ,  $1 \leq p_0 \leq q_0$ , and suppose that  $R^*(p_0 \rightarrow q_0)$  holds.*

*Let  $q_0 < q < \infty$ ,  $q \leq r \leq \infty$  and suppose that  $0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{1}{p_0} - \frac{1}{q_0}$ .*

*Then  $U : B_{\alpha,q}^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d; L^r(I))$  is bounded with  $\alpha = d(1 - \frac{1}{p} - \frac{1}{q}) - \frac{2}{r}$ .*

Using also the trivial  $R^*(1 \rightarrow \infty)$  one can deduce the conclusion in the larger range  $p_1(q) < p \leq q$ , where  $p_1(q) < p_0$  is defined by  $\frac{1}{p_1(q)} = \frac{1}{p_0} + (1 - \frac{q_0}{q})(1 - \frac{1}{p_0})$ .

Given Theorem 1.6 the recent progress on  $R^*(p \rightarrow p)$  by Bourgain and Guth [3] can be used to verify Conjecture 1.5 for new parameters (see also Proposition 5.2 below for the

case  $p \neq q$ ). In two dimensions their result implies that the conjecture holds in the case  $p = q \leq r$  for  $p > 33/10$ ; moreover, in higher dimensions, it holds for  $p > p_{BG}(d)$  with  $p_{BG}(d) = 2 + 3d^{-1} + O(d^{-2})$  (see [3] for their exact range of  $p$ ).

In two dimensions a better range for  $p$  can be obtained for large  $r$ ; this is closely related to previous results on maximal operators for  $L^2_\alpha$  functions and results on Planchon's conjecture in  $\mathbb{R}^2$  (cf. [21], [15], [23], [17]).

**Theorem 1.8.** *Let  $\frac{16}{5} < p < \infty$  and  $4 \leq r \leq \infty$ .*

*Then  $U : B^p_{\alpha,p}(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2; L^r(I))$  is bounded with  $\alpha = 2(1 - \frac{2}{p}) - \frac{2}{r}$ .*

The  $r$ -range can be further improved for  $16/5 < p < 4$ , by interpolating with the above mentioned  $L^p(L^p(I))$  bounds for  $p > 33/10$  ([3]) and the  $L^p(L^2(I))$  bounds in [16] for  $p > 4$ . Moreover one can obtain intermediate  $L^p_\alpha \rightarrow L^q(L^r(I))$  bounds with the critical  $\alpha$  by interpolating with the  $L^2 \rightarrow L^q(L^r)$  bounds in [17].

One can also interpolate with the best known  $L^2(\mathbb{R}^2)$  estimates for the maximal operator  $f \mapsto \sup_{t \in I} |Uf(\cdot, t)|$ , which are equivalent to the best known local estimates (see [14, 23]).

**Corollary 1.9.** *Let  $2 \leq p \leq 16/5$ .*

*Then  $U : L^p_\alpha(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2; L^\infty(I))$  is bounded with  $\alpha > 3/4$ .*

Unlike the rest of the estimates in this article, there is no reason to suspect that this is sharp with respect to the regularity in the range  $2 \leq p < 16/5$ .

*Notation.* By  $m(D)$  we denote the convolution operator with Fourier multiplier  $m$ ; that is to say  $\widehat{m(D)f} = m\widehat{f}$ . For two nonnegative quantities  $A, B$  the notation  $A \lesssim B$  is used for  $A \leq CB$ , with some unspecified constant  $C$ . We also use  $A \approx B$  to indicate that  $A \lesssim B$  and  $B \lesssim A$ .

## 2. $L^p \rightarrow L^q(L^r(I))$ BOUNDS AND THE ADJOINT RESTRICTION OPERATOR

We formulate a more technical version of Theorem 1.6 that applies to mixed norm inequalities. In what follows let

$$(2.1) \quad \mathcal{A}(\rho) := \{\xi \in \mathbb{R}^d : 3\rho \leq |\xi| \leq 12\rho\}.$$

**Theorem 2.1.** *Let  $p, q, r \in [2, \infty]$ ,  $p \leq q$ ,  $\beta > -d(\frac{1}{2} - \frac{1}{p})$ . Then the inequality*

$$(2.2) \quad \sup_{\lambda > 1} \lambda^{-\beta} \sup_{\|f\|_p \leq 1} \left( \int_{\mathcal{A}(\lambda)} \left( \int_\lambda^{2\lambda} |\mathcal{E}f(\frac{s}{\lambda}\xi, s)|^r ds \right)^{q/r} d\xi \right)^{1/q} < \infty$$

*holds if and only if for  $\gamma = d(1 - \frac{1}{p} - \frac{1}{q}) - \frac{2}{r} + 2\beta$ ,*

$$(2.3) \quad \sup_{\|f\|_{B^p_{\gamma,1}} \leq 1} \left\| \left( \int_{-1}^1 |e^{it\Delta} f|^r dt \right)^{1/r} \right\|_q < \infty.$$

Taking Theorem 2.1 for granted we can quickly give

*Proof of Theorem 1.6.* By Theorem 2.1 we just have to show that  $R^*(p \rightarrow q)$  is equivalent with (2.2) for large  $\lambda$ , in the case  $q = r$  and  $\beta = 0$ . Clearly the latter is implied by  $R^*(p \rightarrow q)$ ; this follows by a change of variables  $(\eta, s) = (s\lambda^{-1}\xi, s)$  which has Jacobian bounded above and below in the region where  $s \approx \lambda$ .

*Vice versa,* supposing that (2.2) holds in the case  $q = r$  and  $\beta = 0$ , by the change of variables, we have that  $\mathcal{E} : L^p(\mathbb{R}^d) \rightarrow L^q(W_\lambda)$ , where

$$W_\lambda = \{(\xi, s) : s \in [\lambda, 2\lambda], \quad x \in \mathcal{A}(s)\}.$$

For  $\omega \in \mathbb{R}^{d+1}$  define  $f^\omega(y) = e^{i(\omega, y) - i\omega_{d+1}|y|^2} f(y)$  and observe that  $\mathcal{E}f^\omega = \mathcal{E}f(\cdot - \omega)$ . Thus using a finite number of translations we see that  $\mathcal{E} : L^p(\mathbb{R}^d) \rightarrow L^q(B_\lambda)$ , where  $B_\lambda$  is the ball in  $\mathbb{R}^{d+1}$  of radius  $\lambda$  centred at the origin, and the operator norm is uniformly bounded in  $\lambda$ . Letting  $\lambda \rightarrow \infty$  yields  $R^*(p \rightarrow q)$ .  $\square$

**Lemma 2.2.** *Let  $p, q, r \in [2, \infty]$  with  $p \leq q$  and let  $\lambda \gg 1$ . Suppose that*

$$(2.4) \quad \left( \int_{\mathcal{A}(\lambda^2)} \left( \int_{\lambda^2}^{2\lambda^2} |\mathcal{E}f(\frac{s}{\lambda^2}\xi, s)|^r ds \right)^{q/r} d\xi \right)^{1/q} \leq A \|f\|_p$$

holds. Then, for  $\psi \in C_c^\infty$  with support in  $\{\xi : 4 < |\xi| < 5\}$ ,

$$(2.5) \quad \left\| \left( \int_{1/2}^1 |e^{it\Delta} \psi(\frac{D}{\lambda}) f|^r dt \right)^{1/r} \right\|_q \lesssim A \lambda^\alpha \|f\|_p, \quad \alpha = d(1 - \frac{1}{p} - \frac{1}{q}) - \frac{2}{r}.$$

*Proof.* If  $f_\lambda$  is the characteristic function of a ball of radius  $(100\lambda)^{-2}$  then  $|\mathcal{E}(f_\lambda)(\frac{s}{\lambda^2}\xi, s)| \geq \lambda^{-2d}$  for  $(\xi, s) \in \mathcal{A}(\lambda^2) \times [\lambda^2, 2\lambda^2]$ . The resulting lower bound  $A \geq c\lambda^{2d(-1+1/p+1/q)+2/r}$  (which is far from being sharp) will be used repeatedly to dominate certain error terms which decay fast in  $\lambda$ .

The convolution kernel for  $e^{it\Delta} \psi(\frac{D}{\lambda})$  can be written as

$$K_t^\lambda(x) = \left( \frac{\lambda}{2\pi} \right)^d \int \psi(\xi) e^{-it\lambda^2|\xi|^2 + i\lambda\langle x, \xi \rangle} d\xi.$$

By integration by parts it follows that

$$(2.6) \quad |K_t^\lambda(x)| \leq C_N |x|^{-N}, \quad \text{for } |x| \geq 11\lambda.$$

Hence, by a standard argument, (2.5) reduces to showing the inequality

$$(2.7) \quad \left( \int_{|x| \leq 11\lambda} \left( \int_{1/2}^1 |K_t^\lambda * f|^r dt \right)^{q/r} dx \right)^{1/q} \lesssim A \lambda^\alpha \|f\|_p, \quad \alpha = d - \frac{d}{p} - \frac{d}{q} - \frac{2}{r}$$

for  $f$  supported in the cube of sidelength  $\lambda(2d)^{-1}$  centred at the origin. Indeed, suppose that (2.7) is verified, let  $\mathfrak{Q}_\lambda = \{Q\}$  be a grid of cubes with sidelength  $\lambda(2d)^{-1}$ , and centres  $x_Q$ , and let  $B_Q$  be the ball of radius  $11\lambda$  centred at  $x_Q$ . Then we may estimate the  $L^q(\mathbb{R}^d; L^r([1/2, 1]))$  norm of  $e^{it\Delta} \psi(\frac{D}{\lambda})$  by

$$(2.8) \quad \begin{aligned} & \left( \int \sum_Q \chi_{B_Q}(x) \left( \int_{1/2}^1 |K_t^\lambda * [f\chi_Q](x)|^r dt \right)^{q/r} dx \right)^{1/q} \\ & + \left( \int \sum_Q \chi_Q(x) \left( \int_{1/2}^1 |K_t^\lambda * [f\chi_{\mathbb{R}^d \setminus B_Q}](x)|^r dt \right)^{q/r} dx \right)^{1/q} \end{aligned}$$

by Minkowski's inequality in  $L^r$ . We use the finite overlap of the balls, the translation invariance of the operators and (2.7) to estimate the first term by

$$CA\lambda^\alpha \left( \sum_Q \|f\chi_Q\|_p^q \right)^{1/q} \lesssim CA\lambda^\alpha \|f\|_p$$

where for the last inequality we have used the assumption  $p \leq q$ . For the second term in (2.8) we use (2.6) with  $N > 2d$  and then Young's inequality to bound it by

$$C \left( \int \left[ \int_{|w| \geq 10\lambda} |w|^{-N} |f(x-w)| dw \right]^q dx \right)^{1/q} \lesssim \lambda^{-N+d(1-\frac{1}{p}+\frac{1}{q})} \|f\|_p \lesssim A\lambda^\alpha \|f\|_p.$$

We used the trivial lower bound for  $A$  in the last step.

Our task is now to prove (2.7). We use a stationary phase calculation to see that  $K_t^\lambda = H_t^\lambda + E_t^\lambda$ , where

$$H_t^\lambda(x) = \frac{e^{-i|x|^2/4t}}{(4\pi it)^{d/2}} \sum_{\nu=0}^M \psi_\nu\left(\frac{x}{2\lambda t}\right) \lambda^{-\nu}$$

and

$$|E_\lambda(x, t)| \leq C_L \lambda^{-L}$$

where we choose  $L \gg d$ . For the leading term  $\psi_0 = \psi$ , and the functions  $\psi_\nu$  are obtained by letting certain differential operators act on  $\psi$ ; thus  $\psi_\nu(w) = 0$  for  $|w| \leq 4$  and  $|w| \geq 5$ .

For the error term we use a trivial bound

$$\left( \int_{|x| \leq 11\lambda} \left( \int_{1/2}^1 \left[ \int |E_\lambda(x-y, t)| |f(y)| dy \right]^r dt \right)^{q/r} dx \right)^{1/q} \lesssim \lambda^{d-L} \|f\|_p \lesssim A \lambda^\alpha \|f\|_p.$$

For the oscillatory terms we have to prove the inequality

$$(2.9) \quad \left( \int_{|x| \leq 11\lambda} \left( \int_{1/2}^1 \left| \int \psi_\nu\left(\frac{x-y}{2\lambda t}\right) \exp\left(i\frac{|x-y|^2}{4t}\right) f(y) dy \right|^r dt \right)^{q/r} dx \right)^{1/q} \lesssim A \lambda^\alpha \|f\|_p$$

whenever  $f$  is supported in  $\{|y| \leq \lambda/2\}$ . By a change of variable  $t \mapsto u = 1/t$  (with  $u \approx t \approx 1$ ) and the support properties for  $\psi_\nu$  this follows from

$$(2.10) \quad \left( \int_{\frac{7}{2}\lambda \leq |x| \leq \frac{21}{2}\lambda} \left( \int_1^2 \left| \int_{|y| \leq \lambda/2} \psi_\nu\left(\frac{u(x-y)}{2\lambda}\right) \exp\left(i\frac{u}{4}(|y|^2 - 2\langle x, y \rangle)\right) \right. \right. \\ \left. \left. \times f(y) dy \right|^r du \right)^{q/r} dx \right)^{1/q} \lesssim A \lambda^\alpha \|f\|_p$$

whenever  $f$  is supported in  $\{|y| \leq \lambda/2\}$ . We now use a parabolic scaling in the  $(x, u)$ -variables and set

$$x = \lambda^{-1}w, \quad u = \lambda^{-2}s; \quad y = 2\lambda z.$$

The previous inequality becomes

$$(2.11) \quad \left( \int_{\frac{7}{2}\lambda^2 \leq |w| \leq \frac{21}{2}\lambda^2} \left( \int_{\lambda^2}^{2\lambda^2} \left| \int_{|z| \leq 1} \psi_\nu\left(\frac{sw - 2\lambda^2 sz}{2\lambda^4}\right) \right. \right. \\ \left. \left. \times \exp\left(i(s|z|^2 - \langle \frac{sw}{\lambda^2}, z \rangle)\right) f(2\lambda z) (2\lambda)^d dz \right|^r \frac{ds}{\lambda^2} \right)^{q/r} \frac{dw}{\lambda^d} \right)^{1/q} \lesssim A \lambda^\alpha \|f\|_p.$$

We have the Fourier series expansion  $\psi_\nu(x) = \sum_{\ell \in \mathbb{Z}^d} c_{\ell, \nu} e^{i\langle \ell, x \rangle}$  for  $x \in [-\frac{9}{10}\pi, \frac{9}{10}\pi]^d$  and for each  $\nu$  the Fourier coefficients are rapidly decaying,  $|c_{\ell, \nu}| \leq C_{N, \nu} (1 + |\ell|)^{-N}$ . Thus

$$\psi_\nu\left(\frac{sw - 2\lambda^2 sz}{2\lambda^4}\right) = \sum_{\ell} c_{\ell, \nu} e^{i\lambda^{-4}\langle sw, \ell \rangle / 2} e^{-i\lambda^{-2}s\langle z, \ell \rangle}.$$

Using Minkowski's inequality for the sum and the rapid decay of the Fourier coefficients the previous inequality (2.10) follows from

$$(2.12) \quad \left( \int_{\frac{7}{2}\lambda^2 \leq |w| \leq \frac{21}{2}\lambda^2} \left( \int_{\lambda^2}^{2\lambda^2} \left| \int_{|z| \leq 1} \exp\left(i(s|z|^2 - \langle \frac{s(w+\ell)}{\lambda^2}, z \rangle)\right) f(2\lambda z) dz \right|^r ds \right)^{q/r} dw \right)^{1/q} \\ \lesssim (1 + |\ell|)^M A \lambda^{\alpha - d + \frac{2}{r} + \frac{d}{q}} \|f\|_p.$$

The left hand side is trivially bounded by  $C\lambda^{2/r+2d/q}$  and therefore the displayed inequality holds for  $|\ell| \geq \lambda^2/4$ . If  $|\ell| \leq \lambda^2/4$ , we change variables and see that for (2.12) we only need to show

$$\left( \int_{3\lambda^2 \leq |w| \leq 11\lambda^2} \left( \int_{\lambda^2}^{2\lambda^2} \left| \int_{|z| \leq 1} \exp(i(s|z|^2 - \langle \frac{sw}{\lambda^2}, z \rangle)) g(z) dz \right|^r ds \right)^{q/r} dw \right)^{1/q} \lesssim A\lambda^{\alpha-d+\frac{2}{r}+\frac{d}{q}} \lambda^{d/p} \|g\|_p.$$

The right hand side is just  $A\|g\|_p$ , so that this would follow from (2.4).  $\square$

**Lemma 2.3.** *Let  $p, q, r \in [2, \infty]$  and  $\lambda \gg 1$ . Let  $2 < a_0 < a_1$  and let  $\eta$  be a radial  $C_c^\infty$  function which satisfies  $\eta(\xi) = 1$  for  $\frac{a_0-2}{4} \leq |\xi| \leq 2(a_1+2)$ . Suppose*

$$(2.13) \quad \sup_{\|f\|_p \leq 1} \left\| \left( \int_{1/2}^1 |e^{it\Delta} \eta(\frac{D}{\lambda}) f|^r dt \right)^{1/r} \right\|_q \leq B.$$

Then

$$(2.14) \quad \left( \int_{a_0\lambda^2 \leq |\xi| \leq a_1\lambda^2} \left( \int_{\lambda^2}^{2\lambda^2} |\mathcal{E}f(\frac{s}{\lambda^2}\xi, s)|^r ds \right)^{q/r} d\xi \right)^{1/q} \lesssim B\lambda^{-d+\frac{d}{p}+\frac{d}{q}+\frac{2}{r}} \|f\|_p.$$

*Proof.* In what follows let  $\alpha = d(1 - \frac{1}{p} - \frac{1}{q}) - \frac{2}{r}$ . We begin by observing the lower bound  $B \geq c\lambda^\alpha$  which follows from the example in §3.2.

By a change of variable  $\xi = \lambda x$ ,  $s = \lambda^2 \rho$ ,  $y = 2\lambda z$  we see that (2.14) is equivalent with

$$\left( \int_{a_0\lambda \leq |x| \leq a_1\lambda} \left( \int_1^2 \left| \int_{|y| \leq 2\lambda} f\left(\frac{y}{2\lambda}\right) e^{i(\rho|y|^2/4 - \rho\langle x, y \rangle/2)} dy \right|^2 d\rho \right)^{q/r} dx \right)^{1/q} \leq CB\lambda^{-\alpha} (2\lambda)^d \lambda^{-d/q-2/r} \|f\|_p.$$

By inverting  $t = 1/\rho$  the previous inequality follows from

$$\left( \int_{a_0\lambda \leq |x| \leq a_1\lambda} \left( \int_{1/2}^1 \left| \frac{1}{(4\pi it)^{d/2}} \int_{|y| \leq 2\lambda} g(y) e^{\frac{i|x-y|^2}{4t}} dy \right|^r dt \right)^{q/r} dx \right)^{1/q} \lesssim B\lambda^{-\alpha} \lambda^{d-d/p-2/r} \lambda^{-d/p} \|g\|_p$$

which can be rewritten as

$$(2.15) \quad \left( \int_{a_0\lambda \leq |x| \leq a_1\lambda} \left( \int_{1/2}^1 |e^{it\Delta} g(x)|^r dt \right)^{q/r} dx \right)^{1/q} \lesssim B\|g\|_p,$$

for  $g$  supported in  $\{y : |y| \leq 2\lambda\}$ . By assumption

$$\left( \int_{a_0\lambda \leq |x| \leq a_1\lambda} \left( \int_{1/2}^1 \left| e^{it\Delta} \eta\left(\frac{D}{\lambda}\right) g(x) \right|^r dt \right)^{q/r} dx \right)^{1/q} \leq B\|g\|_p$$

and thus (2.14) follows from the straightforward estimate

$$(2.16) \quad \left( \int_{a_0\lambda \leq |x| \leq a_1\lambda} \left( \int_{1/2}^1 \left| e^{it\Delta} (I - \eta\left(\frac{D}{\lambda}\right)) g(x) \right|^r dt \right)^{q/r} dx \right)^{1/q} \leq C_M \lambda^{-M} \|g\|_p,$$

whenever  $g$  is supported in  $\{y : |y| \leq 2\lambda\}$ .

To see (2.16) we decompose the multiplier. Let  $\chi_0$  be smooth and supported in  $\{|\xi| < 2\}$  and  $\chi_0(\xi) = 1$  for  $|\xi| \leq 1$ , and let  $\chi_k(\xi) = \chi_0(2^{-k}\xi) - \chi_0(2^{1-k}\xi)$ , for  $k \geq 1$ . Let

$$E_{\lambda, k}(x, t) = \frac{1}{(2\pi)^d} \int \chi_k\left(\frac{\xi}{\lambda}\right) (1 - \eta\left(\frac{\xi}{\lambda}\right)) e^{-it|\xi|^2 + i\langle x, \xi \rangle} d\xi$$

and we need to bound the expression

$$(I - \eta(\frac{D}{\lambda}))e^{it\Delta}g(x, t) = \sum_{k \geq 0} \int_{|y| \leq 2\lambda} E_{\lambda, k}(x - y)g(y)dy.$$

We now examine  $\nabla_{\xi}(\langle x - y, \xi \rangle - t|\xi|^2) = x - y - 2t\xi$ . Since  $a_0 > 2$ , for the relevant choices  $a_0|\lambda| \leq |x| \leq a_1\lambda$ ,  $1/2 \leq t \leq 1$ ,  $|y| \leq 2\lambda$  we have

$$|x - y - 2t\xi| \geq \begin{cases} \frac{1}{2}(a_0 - 2)\lambda & \text{if } |\xi| \leq \frac{a_0 - 2}{4}\lambda, \\ \max\{\frac{|\xi|}{2}, (a_1 + 2)\lambda\} & \text{if } |\xi| \geq (a_1 + 2)\lambda. \end{cases}$$

Since  $1 - \eta(\lambda^{-1}\xi) = 0$  for  $\frac{a_0 - 2}{4} \leq |\xi| \leq 2(a_1 + 2)$ , after an  $N$ -fold integration by parts we find that  $|E_{\lambda, k}(x - y, t)| \leq C_N(2^k\lambda)^{d-N}$  for this choice of  $x, y, t$ , and the estimate (2.16) follows.  $\square$

To complete the proof of Theorem 2.1 we also need the following scaling lemma.

**Lemma 2.4.** *Let  $\gamma > d(\frac{1}{p} - \frac{1}{q}) - \frac{2}{r}$ . Suppose that for  $\lambda \gg 1$*

$$(2.17) \quad \left\| \left( \int_{1/2}^1 |e^{it\Delta} \chi(\frac{D}{\lambda}) f|^r dt \right)^{1/q} \right\|_q \lesssim \lambda^\gamma \|f\|_p,$$

where  $\chi \in C_c^\infty$  is supported in  $(1/2, 2)$  (with suitable bounds). Then, for  $\lambda \gg 1$ ,

$$(2.18) \quad \left\| \left( \int_I |e^{it\Delta} \chi(\frac{D}{\lambda}) f|^r dt \right)^{1/r} \right\|_q \lesssim \lambda^\gamma \|f\|_p.$$

*Proof.* It is easy to calculate that

$$\sup_{0 \leq t \leq (8\lambda)^{-2}} |\mathcal{F}^{-1}[\chi(\frac{\cdot}{\lambda}) \exp(-it|\cdot|^2)](x)| \leq C_N \lambda^d (1 + \lambda|x|)^{-N}$$

and thus, by Young's inequality,

$$(2.19) \quad \begin{aligned} \left\| \left( \int_0^{(8\lambda)^{-2}} |e^{it\Delta} \chi(\frac{D}{\lambda}) f|^r dt \right)^{1/r} \right\|_q &\lesssim \left\| \lambda^{-2/r} \int \lambda^d (1 + \lambda|y|)^{-N} |f(\cdot - y)| dy \right\|_q \\ &\lesssim \lambda^{d(\frac{1}{p} - \frac{1}{q}) - \frac{2}{r}} \|f\|_p. \end{aligned}$$

Now letting  $(8\lambda)^{-2} \leq b \leq 1$ ,

$$\left( \int_{b/2}^b |e^{it\Delta} \chi(\frac{D}{\lambda}) f(x)|^r dt \right)^{1/r} = b^{1/r} \left( \int_{1/2}^1 \left| \chi(\frac{D}{b^{1/2}\lambda}) e^{is\Delta} [f(b^{1/2}\cdot)](b^{-1/2}x) \right|^r ds \right)^{1/r}.$$

Thus by a change of variable (2.17) implies

$$\left\| \left( \int_{b/2}^b |e^{it\Delta} \chi(\frac{D}{\lambda}) f|^r dt \right)^{1/r} \right\|_q \lesssim (\sqrt{b})^{-d(\frac{1}{p} - \frac{1}{q}) + \frac{2}{r}} (\lambda\sqrt{b})^\gamma \|f\|_p.$$

We choose  $b = 2^{-l}$ , and since  $\gamma > d(\frac{1}{p} - \frac{1}{q}) - \frac{2}{r}$  we may sum over  $l$  with  $(8\lambda)^{-2} \leq 2^{-l} \leq 1$  and combine with (2.19). Hence we get

$$\left\| \left( \int_0^1 |e^{it\Delta} \chi(\frac{D}{\lambda}) f|^r dt \right)^{1/r} \right\|_q \lesssim \lambda^\gamma \|f\|_p.$$

Now (2.18) with  $I = [-1, 1]$  follows using the formula  $e^{-it\Delta} f = \overline{e^{it\Delta} \overline{f}}$ , and the triangle inequality. Finally, by scaling, we can enlarge the time interval (so that the implicit constant is of course dependent on the interval), and we are done.  $\square$

*Proof of Theorem 2.1.* The implication (2.3)  $\Rightarrow$  (2.2) follows from Lemma 2.3. Now if  $\beta > -d(1/2 - 1/p)$  then the restriction on  $\gamma$  in Lemma 2.4 is satisfied for the value  $\gamma = d(1 - 1/p - 1/q) - 2/r + 2\beta$ . The implication (2.2)  $\Rightarrow$  (2.3) follows by combining Lemmata 2.2 and 2.4 (together with dyadic and finite decompositions, and mild rescaling).  $\square$

### 3. PROOF OF PROPOSITION 1.1

First we discuss the easier necessary conditions (i)-(iv).

3.1. *The condition  $p \leq q$ .* This follows from the translation invariance (see an argument in [11]). More precisely, the  $L^p_\alpha(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d; L^r(I))$  boundedness is equivalent with the  $L^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d; L^r(I))$  boundedness of the operator  $U[(I - \Delta)^{\alpha/2}f]$  which commutes with translations on  $\mathbb{R}^d$ . Let  $A = \sup_{\|f\|_p \leq 1} \|U[(I - \Delta)^{\alpha/2}f]\|_{L^q(L^r)}$ . Then by the density argument, for  $\epsilon > 0$  there is a  $g \in C_c^\infty(\mathbb{R}^d)$  such that  $A - \epsilon < \|U[(I - \Delta)^{\alpha/2}g]\|_{L^q(L^r)}$  and  $\|g\|_p = 1$ . One may test the inequality with  $f = g + g(\cdot + ae_1)$ . Letting  $a \rightarrow \infty$ , we see that  $(A - \epsilon)2^{1/q} \leq A2^{1/p}$ , which gives  $A2^{1/q} \leq A2^{1/p}$  by letting  $\epsilon \rightarrow 0$ , and thus  $p \leq q$ .

3.2. *The condition  $\alpha \geq d(1 - \frac{1}{p} - \frac{1}{q}) - \frac{2}{r}$ .* This condition follows by a focusing example (see for example [22]). Let  $\eta \in C_c^\infty$  be radial and supported in  $\{\xi : 1 < |\xi| < 2\}$ . Define for  $\lambda \gg 1$ , the function  $f_\lambda$  by  $\widehat{f}_\lambda(\xi) = e^{i\frac{1}{2}|\xi|^2} \eta(\lambda^{-1}\xi)$ . Then  $\|f_\lambda\|_{L^p_\alpha} \lesssim \lambda^{\alpha+d/p}$ . Moreover  $|Uf(x, t)| \gtrsim \lambda^d$  if, for suitable  $c > 0$ ,  $|x| \leq c\lambda^{-1}$  and  $|t - \frac{1}{2}| \leq c\lambda^{-2}$ . For large  $\lambda$  this leads to the restriction  $\alpha \geq d(1 - \frac{1}{p} - \frac{1}{q}) - \frac{2}{r}$ .

3.3. *The condition  $\alpha \geq \frac{1}{q} - \frac{1}{r}$ .* Let  $g_\lambda$  be defined by  $\widehat{g}_\lambda(\xi) = \chi(|\xi - \lambda e_1|)$ ,  $\chi$  supported in an  $\epsilon$ -neighborhood of 0 (see [7], [24]), so that  $\|g_\lambda\|_{L^p_\alpha} \lesssim \lambda^\alpha$ . Also,

$$Ug_\lambda(x, t) = \frac{1}{(2\pi)^d} \int \chi(|h|) e^{i\phi_\lambda(x, t, h)} dh$$

where  $\phi_\lambda(x, t, h) = -t|h|^2 - t\lambda^2 + x_1\lambda + \langle x - 2t\lambda e_1, h \rangle$ . Then  $|Ug_\lambda(x, t)| \geq c_0 > 0$  if  $|t - (2\lambda)^{-1}x_1| \leq c\lambda^{-1}$  for  $0 \leq x_1 \leq \lambda$ ,  $|x_i| \leq c$ ,  $i = 2, \dots, d$ . It follows that  $\|Uf\|_{L^q(L^r(I))} \geq \lambda^{1/q-1/r}$ . Hence the condition  $\alpha \geq 1/q - 1/r$  follows.

3.4. *The condition  $\alpha \geq \frac{1}{q} - \frac{1}{p}$ .* Let  $\lambda \gg 1$  and set  $\widehat{h}_\lambda(\eta) = \phi(|\eta'|)\lambda\phi(\lambda(\eta_1 - \lambda))$  with  $\phi \in C_c^\infty(\mathbb{R})$ . Then  $\|h_\lambda\|_{L^p_\alpha} \lesssim \lambda^\alpha \lambda^{1/p}$ . Note that

$$Uh_\lambda(x, t) = \frac{1}{(2\pi)^d} \int e^{-it|\eta'|^2 + i\langle x', \eta' \rangle} \phi(|\eta'|) d\eta' e^{-i\lambda^2 t + i\lambda x_1} \int e^{i(-t\xi_1^2 - 2\lambda t\xi_1 + x_1\xi_1)} \lambda\phi(\lambda\xi_1) d\xi_1,$$

so that  $|Uh_\lambda(x, t)| \geq c > 0$  if  $|t|, |x'| \leq c$  and  $|x_1| \leq c\lambda$  for small enough  $c > 0$ . This shows the necessity of  $\alpha \geq 1/q - 1/p$ .

To show the conditions (v) and (vi), we use sharp bounds in the construction of Besicovich sets [13] and adapt Fefferman's argument for the disc multiplier [9] (see also [1]).

3.5. *The condition  $\alpha > \frac{1}{q} - \frac{1}{p}$  if  $r > 2$ .* This follows from

**Proposition 3.1.** *Let  $p, q, r \in [2, \infty)$ . Let  $\eta$  be a radial  $C_c^\infty$  function satisfying  $\eta(\xi) = 1$  for  $1/4 \leq |\xi| \leq 12$ . Define  $\mathbf{a}_\lambda$  by*

$$(3.1) \quad \mathbf{a}_\lambda(p; q, r) = \sup_{\|f\|_p \leq 1} \left\| \left( \int_{1/2}^1 |e^{it\Delta} \eta(\frac{D}{\lambda}) f|^r dt \right)^{1/r} \right\|_{L^q(\mathbb{R}^d)}.$$

Then for  $\lambda \gg 1$ ,

$$(3.2) \quad \mathbf{a}_\lambda(p; q, r) \geq c\lambda^{1/q-1/p}(\log \lambda)^{1/2-1/r}.$$

*Proof.* In what follows we set

$$\mathcal{A}_4(\lambda^2) = \{x : 3\lambda^2 \leq |\xi| \leq 4\lambda^2\}.$$

By Lemma 2.3, with parameters  $a_0 = 3$ ,  $a_1 = 4$ , for  $\lambda \gg 1$

$$\sup_{\|f\|_{L^p(\mathbb{R}^d)} \leq 1} \left( \int_{\mathcal{A}_4(\lambda^2)} \left( \int_{\lambda^2}^{2\lambda^2} |\mathcal{E}f(\frac{s}{\lambda^2}\xi, s)|^r ds \right)^{\frac{q}{r}} d\xi \right)^{\frac{1}{q}} \lesssim \mathbf{a}_\lambda(p; q, r) \lambda^{-d+\frac{d}{p}+\frac{d}{q}+\frac{2}{r}}.$$

Let

$$Tf(\xi, s) = \mathcal{E}f(\frac{s}{\lambda^2}\xi, s).$$

Using Khintchine's inequality we also get

$$(3.3) \quad \sup_{\{\|f_j\|\}_{L^p(\ell^2)} \leq 1} \left( \int_{\mathcal{A}_4(\lambda^2)} \left( \int_{\lambda^2}^{2\lambda^2} \left( \sum_j |Tf_j|^2 \right)^{\frac{r}{2}} ds \right)^{\frac{q}{r}} d\xi \right)^{\frac{1}{q}} \lesssim \mathbf{a}_\lambda(p; q, r) \lambda^{-d+\frac{d}{p}+\frac{d}{q}+\frac{2}{r}}.$$

For integers  $|j| \leq \lambda/10$ , let  $z^j = (\lambda^{-1}j, 0, \dots, 0)$  in  $\mathbb{R}^d$ . Let  $I_j = \{y : |y - z^j| \leq (100d\lambda)^{-1}\}$ . Let

$$R_j = \{(\xi, s) \in \mathbb{R}^{d+1} : |\xi_1 - 2j\lambda^{-1}s| \leq 10^{-1}\lambda, |\xi_i| \leq 10^{-1}\lambda, \quad i = 2, \dots, d, |s| \leq 100^{-1}\lambda^2\}.$$

For a pointwise lower bound we use the following lemma.

**Lemma 3.2.** *Let  $a \in \mathbb{R}^d$ ,  $b \in \mathbb{R}$ , and  $g_j(y) = \chi_{I_j}(y)e^{i\langle a, y \rangle - ib|y|^2}$ . Then there is a constant  $c > 0$ , independent of  $\lambda, j$  so that*

$$\operatorname{Re} [e^{i\langle \xi - a, z^j \rangle - i(s-b)|z^j|^2} \mathcal{E}[g_j](\xi, s)] \geq c\lambda^{-d}, \quad \text{if } (\xi, s) \in R_j + (a, b).$$

*Proof.* Let  $I_0 = \{y : |y| \leq (100d\lambda)^{-1}\}$ . We have

$$\begin{aligned} \mathcal{E}g_j(\xi, s) &= \int e^{is|y|^2 - i\langle \xi, y \rangle} g_j(y) dy = \int e^{-i\langle \xi - a, z^j + h \rangle + i(s-b)|z^j + h|^2} \chi_{I_j}(z^j + h) dh \\ &= e^{-i\langle \xi - a, z^j \rangle} e^{i(s-b)|z^j|^2} \int e^{-i(\langle \xi - a - 2(s-b)z^j, h \rangle)} e^{i(s-b)|h|^2} \chi_{I_0}(h) dh \end{aligned}$$

The pointwise lower bound follows quickly.  $\square$

Let  $N_\lambda$  to be the largest integer which is smaller than  $\lambda/10$ . By making use of the Besicovich set construction of Keich [13], there are vectors  $v_j \in \mathbb{R}^{d+1}$  such that  $v_j = a_j e_1 + b_j e_{d+1}$  for some  $a_j, b_j \in \mathbb{R}$ ,  $v_j + R_j \subset \{(\xi, s) : \lambda^2 \leq s \leq 2\lambda^2\}$ , and

$$\operatorname{meas} \left( \bigcup_{j=1}^{N_\lambda} (v_j + R_j) \right) \lesssim \frac{\lambda^{d+3}}{\log \lambda}.$$

This is just an obvious extension of the two dimensional construction which gives a collection of rectangles  $\{R_j^{[2]}\}$  and vectors  $(a_j, b_j)$  such that  $\operatorname{meas}(\bigcup_{j=1}^{N_\lambda} (v_j + R_j^{[2]})) \lesssim \frac{\lambda^4}{\log \lambda}$  and  $(a_j, b_j) + R_j^{[2]} \subset \{(\xi_1, s) : \lambda^2 \leq s \leq 2\lambda^2\}$ .

Let  $\Phi(\xi, s) = (\frac{s}{\lambda^2}\xi, s)$  which is 1-1 on  $\mathcal{A}_4(\lambda^2) \times [\lambda^2, 2\lambda^2]$ , and has Jacobian  $J_\Phi$  with  $|\det(J_\Phi(\xi, s))| \sim 1$ . Let

$$V_j := \Phi^{-1}(v_j + R_j) \cap (\mathcal{A}_4(\lambda^2) \times [\lambda^2, 2\lambda^2]), \quad E := \bigcup_{j=1, \dots, N_\lambda} V_j.$$

Then it follows that

$$(3.4) \quad \lambda^{d+2} \lesssim \text{meas}(V_j), \quad \text{meas}(E) \lesssim \frac{\lambda^{d+3}}{\log \lambda}.$$

Let  $f_j(y) = \chi_{I_j}(y)e^{i\langle a_j, y \rangle - ib_j|y|^2}$ . Then by Lemma 3.2,

$$(3.5) \quad |Tf_j(\xi)| \gtrsim \lambda^{-d}, \quad \xi \in V_j,$$

and

$$(3.6) \quad \left\| \left( \sum |f_j|^2 \right)^{1/2} \right\|_p \lesssim \lambda^{(1-d)/p}.$$

We now modify arguments in [1]. By (3.4) and (3.5), we have

$$(3.7) \quad \begin{aligned} \lambda^{d+3} &\lesssim N_\lambda \lambda^{d+2} \lesssim \sum_{j=1}^{N_\lambda} \text{meas}(V_j) \\ &= \int_E \sum_{j=1}^{N_\lambda} \chi_{V_j}(\xi, s) ds d\xi \lesssim \lambda^{2d} \int_E \sum_{j=1}^{N_\lambda} |Tf_j(\xi, s)|^2 ds d\xi, \end{aligned}$$

and by applications of Hölder's inequality,

$$(3.8) \quad \lambda^{2d} \int_E \sum_{j=1}^{N_\lambda} |Tf_j(\xi, s)|^2 ds d\xi \lesssim \lambda^{2d} A \cdot B,$$

where

$$\begin{aligned} A &= \left( \int_{\mathcal{A}_4(\lambda^2)} \left( \int_{\lambda^2}^{2\lambda^2} \left( \sum_j |Tf_j(\xi, s)|^2 \right)^{\frac{r}{2}} ds \right)^{\frac{q}{r}} d\xi \right)^{\frac{2}{q}}, \\ B &= \left( \int_{\mathcal{A}_4(\lambda^2)} \left( \int_{\lambda^2}^{2\lambda^2} \chi_E(\xi, s) ds \right)^{\frac{(q/2)'}{(r/2)'}} d\xi \right)^{1-\frac{2}{q}}. \end{aligned}$$

From (3.3) and (3.6) we obtain,

$$(3.9) \quad A \lesssim \left( \lambda^{\frac{1-d}{p}} \mathbf{a}_\lambda(p; q, r) \lambda^{-d+\frac{1}{p}+\frac{d}{q}+\frac{2}{r}} \right)^2.$$

In order to estimate  $B$  we set

$$\mathbf{v}(\xi) = \int_{\lambda^2}^{2\lambda^2} \chi_E(\xi, s) ds,$$

the measure of the vertical cross section of  $E$  at  $\xi$ . For  $M > 0$ , we break

$$B \lesssim \left( \int_{\{\xi \in \mathcal{A}_4(\lambda^2) : \mathbf{v}(\xi) \leq M\}} \mathbf{v}(\xi)^{\frac{(q/2)'}{(r/2)'}} d\xi \right)^{1-\frac{2}{q}} + \left( \int_{\{\xi \in \mathcal{A}_4(\lambda^2) : \mathbf{v}(\xi) > M\}} \mathbf{v}(\xi)^{\frac{(q/2)'}{(r/2)'}} d\xi \right)^{1-\frac{2}{q}}.$$

From the construction of  $E$  it is obvious that  $\mathbf{v}$  is supported in a tube where  $|\xi_1| \leq C\lambda^2$  and  $|\xi_i| \leq C\lambda$ ,  $2 \leq i \leq d$ , so that

$$\left( \int_{\{\xi \in \mathcal{A}_4(\lambda^2) : \mathbf{v}(\xi) \leq M\}} \mathbf{v}(\xi)^{\frac{(q/2)'}{(r/2)'}} d\xi \right)^{1-\frac{2}{q}} \lesssim M^{1-\frac{2}{r}} \lambda^{(d+1)(1-\frac{2}{q})}.$$

Moreover since  $r \leq q$  and therefore  $(1 - \frac{(q/2)'}{(r/2)'}) \geq 0$ , by (3.4)

$$\begin{aligned} \left( \int_{\{\xi \in \mathcal{A}_4(\lambda^2): \mathbf{v}(\xi) > M\}} \mathbf{v}(\xi)^{\frac{(q/2)'}{(r/2)'}} d\xi \right)^{1-\frac{2}{q}} &\lesssim \left( \int \mathbf{v}(\xi) M^{\frac{(q/2)'}{(r/2)'}-1} d\xi \right)^{1-\frac{2}{q}} \\ &\leq M^{\frac{2}{q}-\frac{2}{r}} \text{meas}(E)^{1-\frac{2}{q}} \lesssim M^{\frac{2}{q}-\frac{2}{r}} \left( \frac{\lambda^{d+3}}{\log \lambda} \right)^{1-\frac{2}{q}}. \end{aligned}$$

Combining these two bounds, we have

$$B \lesssim M^{-2/r} \lambda^{(d+3)(1-\frac{2}{q})} [M \lambda^{-2(1-\frac{2}{q})} + M^{\frac{2}{q}} (\log \lambda)^{\frac{2}{q}-1}],$$

and choosing  $M = \lambda^2 (\log \lambda)^{-1}$ , which optimizes the above, we obtain

$$(3.10) \quad B \lesssim \lambda^{(d+3)(1-\frac{2}{q})} \lambda^{\frac{4}{q}-\frac{4}{r}} (\log \lambda)^{\frac{2}{r}-1}.$$

Finally, we combine (3.10), (3.9), (3.8) and (3.7) to obtain

$$\lambda^{d+3} \lesssim \lambda^{2d} \lambda^{(d+3)(1-\frac{2}{q})} \lambda^{\frac{4}{q}-\frac{4}{r}} (\log \lambda)^{\frac{2}{r}-1} \left[ \lambda^{\frac{1-d}{p}} \mathbf{a}_\lambda(p; q, r) \lambda^{-d+\frac{d}{p}+\frac{d}{q}+\frac{2}{r}} \right]^2,$$

which yields  $\mathbf{a}_\lambda(p; q, r) \geq c (\log \lambda)^{\frac{1}{2}-\frac{1}{r}} \lambda^{\frac{1}{q}-\frac{1}{p}}$ .  $\square$

3.6. *Relation with Bochner–Riesz and the condition  $\alpha > 0$  if  $r = 2$ ,  $p = q > 2$ ,  $d \geq 2$ .* The  $L^p \rightarrow L^p(L^2(I))$  estimate implies sharp results for the Bochner–Riesz multiplier in the same way as the wave equation (cf. §7 in [19]).

For small  $\delta > 0$ , let us set  $h_\delta(\xi) = \phi(\delta^{-1}(1 - |\xi|^2))$  with  $\phi \in C_c^\infty(-1, 1)$ . Let  $\psi$  be radial, supported in  $\{1/2 < |\xi| < 2\}$  so that  $\psi = 1$  on the support of  $h_\delta$ . Then by the Fourier inversion formula and the support property of  $\psi$  it follows that

$$h_\delta(D)f = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta \widehat{\phi}(\delta s) e^{is} e^{is\Delta} \psi(D)f ds.$$

By the Schwarz inequality we get

$$|h_\delta(D)f| \leq \left( \int |\delta \widehat{\phi}(\delta s)| ds \right)^{1/2} \left( \int |e^{is\Delta} \psi(D)f|^2 |\delta \widehat{\phi}(\delta s)| ds \right)^{1/2}.$$

Thus we see that

$$\|h_\delta\|_{M_p} \lesssim \sup_{\|f\|_p \leq 1} \left\| \left( \int |e^{is\Delta} \psi(D)f|^2 |\delta \widehat{\phi}(\delta s)| ds \right)^{1/2} \right\|_p,$$

which after rescaling becomes

$$\|h_\delta\|_{M_p} \lesssim \sup_{\|f\|_p \leq 1} \left\| \left( \int |e^{it\Delta} \psi(\sqrt{\delta}D)f|^2 |\widehat{\phi}(t)| dt \right)^{1/2} \right\|_p.$$

Hence, using the rapid decay of  $\widehat{\phi}$  and a further rescaling we see that the sharp bound  $\|h_\delta\|_{M_p} \lesssim \delta^{1/2-d(1/2-1/p)}$ , for  $p > 2 + \frac{2}{d-1}$ , would follow from  $U : B_{\alpha, \nu}^p \rightarrow L^p(L^2(I))$ , with  $\alpha = d(1 - \frac{2}{p}) - 1$ , for any  $\nu > 0$ .

We see that the  $L^p \rightarrow L^p(L^2(I))$  inequality for some  $p > 2$  would imply that  $h_\delta$  is a multiplier of  $\mathcal{FL}^p$  with bounds independent of  $\delta$ . However a variant of Fefferman's argument for the ball multiplier [9], based on a Kakeya set argument, shows that

$$(3.11) \quad \|h_\delta\|_{M_p} \gtrsim \log(1/\delta)^{1/2-1/p}.$$

This establishes the final necessary condition (vi) in Proposition 1.1. For completeness we include some details of the argument.

*Proof of (3.11).* By de Leeuw's theorem it suffices to prove the lower bound for  $d = 2$ . We may assume that  $\delta < 10^{-10}$ . By Khintchine's inequality, we have

$$(3.12) \quad \left\| \left( \sum_{\nu} |h_{\delta}(D)f_{\nu}|^2 \right)^{1/2} \right\|_p \lesssim \|h_{\delta}\|_{M_p} \left\| \left( \sum_{\nu} |f_{\nu}|^2 \right)^{1/2} \right\|_p.$$

For  $\nu \in \mathbb{Z} \cap [-10^{-2}\delta^{-1/2}, 10^{-2}\delta^{-1/2}]$ , let us set

$$h_{\delta,\nu}(\xi) = h_{\delta}(\xi)\phi(\delta^{-1/2}\xi_1 - \nu)\chi_+(\xi), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2$$

where  $\chi_+$  is the characteristic function of the upper half plane. Define  $T_{\nu}$  by  $\widehat{T_{\nu}f} = h_{\delta,\nu}\widehat{f}$ . Let  $\eta_{\nu}$  be the inverse Fourier transform of a bump function which is supported on a ball of radius  $C\delta^{-1/2}$  so that  $\eta_{\nu}(\xi) = 1$  for  $\xi$  in the support of  $h_{\delta,\nu}$ . Define  $\Phi_{\nu}$  by  $\widehat{\Phi_{\nu}}(\xi) = \eta_{\nu}(\xi)\phi(\delta^{-1/2}\xi_1 - \nu)\chi_+(\xi)$ . Then  $|\Phi_{\nu}(x)| \lesssim \delta^{-d/2}(1 + \delta^{-1/2}|x|)^{-(d+1)}$  for the  $\nu$ 's under consideration, so that  $\|\{\Phi_{\nu} * g_{\nu}\}\|_{L^p(\ell^2)} \lesssim \|\{g_{\nu}\}\|_{L^p(\ell^2)}$ . Since  $T_{\nu}g = h_{\delta}(D)[\Phi_{\nu} * g]$ , inequality (3.12) applied to  $f_{\nu} = \Phi_{\nu} * g_{\nu}$  implies that

$$(3.13) \quad \left\| \left( \sum_{\nu} |T_{\nu}g_{\nu}|^2 \right)^{1/2} \right\|_p \lesssim \|h_{\delta}\|_{M_p} \left\| \left( \sum_{\nu} |g_{\nu}|^2 \right)^{1/2} \right\|_p.$$

Let  $\theta_{\nu} = (\delta^{1/2}\nu, \sqrt{1 - \delta\nu^2})$ , let  $\theta_{\nu}^{\perp}$  be a unit vector perpendicular to  $\theta_{\nu}$  and

$$R_{\nu} = \{(x_1, x_2) : |\langle x, \theta_{\nu} \rangle| \leq 10^{-2}\delta^{-1}, |\langle x, \theta_{\nu}^{\perp} \rangle| \leq 10^{-1}\delta^{-1/2}\}.$$

Letting  $f_{\nu}(y) = \chi_{R_{\nu}}(y)e^{i\langle \theta_{\nu}, y \rangle}$ , we have that

$$(3.14) \quad |e^{-i\langle x, \theta_{\nu} \rangle} T_{\nu}f_{\nu}(x)| \geq c > 0 \text{ for } x \in R_{\nu}.$$

Here we use again the sharp bounds in the construction of Besicovich sets [13]. There are vectors  $a_{\nu}$ ,  $|\nu| \leq 10^{-2}\delta^{-1/2}$  so that with  $E := \bigcup_{\nu} R_{\nu}$  the measure of  $E$  is  $O(\delta^{-2}/\log \delta^{-1})$  but the corresponding translations  $a_{\nu} + R_{\nu}$  have  $O(1)$  overlap. Define  $g_{\nu}(x) = f_{\nu}(x - a_{\nu})$ , which is supported in  $a_{\nu} + R_{\nu}$ . Then  $|T_{\nu}g_{\nu}| \geq c$  on  $a_{\nu} + R_{\nu}$ . Thus we get

$$\delta^{-2} \lesssim \sum_{\nu} |R_{\nu}| \lesssim \sum_{\nu} \int \chi_{a_{\nu} + R_{\nu}}(x) dx \lesssim \int_E \sum_{\nu} |T_{\nu}g_{\nu}|^2 dx$$

and also by Hölder's inequality and (3.13) the last one in the above string of inequalities is bounded by

$$\text{meas}(E)^{1-2/p} \left\| \left( \sum_{\nu} |T_{\nu}g_{\nu}|^2 \right)^{1/2} \right\|_p^2 \lesssim \|h_{\delta}\|_{M_p}^2 \left( \frac{\delta^{-2}}{\log \delta^{-1}} \right)^{1-2/p} \left\| \left( \sum_{\nu} |g_{\nu}|^2 \right)^{1/2} \right\|_p^2.$$

Now by the bounded overlap of the translated rectangles  $a_{\nu} + R_{\nu}$ , we see

$$\left\| \left( \sum_{\nu} |g_{\nu}|^2 \right)^{1/2} \right\|_p^2 \lesssim \left( \int \sum_{\nu} \chi_{a_{\nu} + R_{\nu}} dx \right)^{2/p} \lesssim \left( \sum_{\nu} |R_{\nu}| \right)^{2/p} \lesssim \delta^{-4/p}.$$

Combining the three displayed inequalities we get  $\delta^{-2} \lesssim \|h_{\delta}\|_{M_p}^2 (\delta^{-2}/\log \delta^{-1})^{1-2/p} \delta^{-4/p}$  and thus the desired (3.11).  $\square$

#### 4. PROOF OF THEOREM 1.3

The lower bounds for  $\mathfrak{A}_{\lambda}(p; q, r)$  were established in the previous section, and here we prove the upper bounds, mainly by interpolation arguments. By Lemma 2.4, we can take  $I = [1/2, 1]$ .

4.1. *Proof of (i).* We consider the cases  $\frac{1}{q} + \frac{1}{r} \geq \frac{1}{2}$  and  $\frac{1}{q} + \frac{1}{r} < \frac{1}{2}$  separately.

The case  $\frac{1}{q} + \frac{1}{r} \geq \frac{1}{2}$ . Note that the set

$$\left\{ \left( \frac{1}{p}, \frac{1}{q}, \frac{1}{r} \right) : 2 \leq r \leq p \leq q \leq \infty, \quad \frac{1}{q} + \frac{1}{r} \geq \frac{1}{2} \right\}$$

is the closed tetrahedron with vertices  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ ,  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ ,  $(\frac{1}{2}, 0, \frac{1}{2})$ , and  $(0, 0, \frac{1}{2})$ . Hence by interpolation it is enough to show the estimate

$$(4.1) \quad \mathfrak{A}_\lambda(p; q, r) \lesssim \lambda^{\frac{1}{q} - \frac{1}{p}} [\log \lambda]^{\frac{1}{2} - \frac{1}{r}}$$

for  $(p, q, r) = (4, 4, 4)$ ,  $(2, 2, 2)$ ,  $(2, \infty, 2)$  and  $(\infty, \infty, 2)$ . The estimate for  $(p, q, r) = (2, 2, 2)$  is immediate from Plancherel's theorem. More generally we recall from [16] the estimate  $\mathfrak{A}_\lambda(p; p, 2) \lesssim 1$  with  $2 \leq p \leq \infty$ , which is related to a square-function estimate for equally spaced intervals. So we also get the estimates for  $(p, q, r) = (\infty, \infty, 2)$ . For  $(2, \infty, 2)$  we choose a nonnegative  $\chi_o \in C_c^\infty(\mathbb{R})$ , so that  $\chi_o(t) = 1$  on  $[1/2, 1]$ . We need to estimate, for fixed  $x$ ,

$$\int \chi_o(t) |U\eta(\frac{D}{\lambda})f(x, t)|^2 dt = \frac{1}{(2\pi)^{2d}} \iint e^{ix(\xi-w)} \widehat{f}(\xi) \overline{\widehat{f}(w)} \eta(\frac{\xi}{\lambda}) \overline{\eta(\frac{w}{\lambda})} \widehat{\chi_o}(|\xi|^2 - |w|^2) d\xi dw$$

and since  $|\xi| + |w| \geq \lambda$ , the above is bounded by

$$C_N \int_{\mathbb{R}} \int_{\mathbb{R}} (1 + \lambda||\xi| - |w||)^{-N} |\widehat{f}(\xi)| |\widehat{f}(w)| d\xi dw \lesssim \lambda^{-1} \|\widehat{f}\|_2^2.$$

This gives the desired estimate for  $(p, q, r) = (2, \infty, 2)$ . For  $(p, q, r) = (4, 4, 4)$  we use the bound

$$\left( \iint \left| \psi(\xi, s) \int_{|y| \leq 1} f(y) e^{i\lambda(s|y|^2 - \xi y)} f(y) dy \right|^4 d\xi ds \right)^{1/4} \lesssim \lambda^{-\frac{1}{2}} (\log \lambda)^{\frac{1}{4}} \|f\|_4,$$

where  $\psi \in C_c^\infty$ . This is implicit in [12] (see also [20] for more discussion and related issues). Then by rescaling, Lemma 2.2 and Lemma 2.4 we get (4.1) for  $(p, q, r) = (4, 4, 4)$ .

The case  $\frac{1}{q} + \frac{1}{r} < \frac{1}{2}$ . We begin as before by observing that the set

$$\Delta_1 = \left\{ \left( \frac{1}{p}, \frac{1}{q}, \frac{1}{r} \right) : 2 \leq r \leq p \leq q \leq \infty, \quad \frac{1}{q} + \frac{1}{r} < \frac{1}{2} \right\}$$

is the closed tetrahedron with vertices  $(0, 0, 0)$ ,  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ ,  $(\frac{1}{2}, 0, \frac{1}{2})$  and  $(0, 0, \frac{1}{2})$ , from which the triangle with vertices  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ ,  $(\frac{1}{2}, 0, \frac{1}{2})$ , and  $(0, 0, \frac{1}{2})$  is removed. We use a bilinear analogue of our adjoint restriction operator, and rely on rather elementary estimates from [12]. Define  $\chi_\ell$  so that  $\sum_{\ell \in \mathbb{Z}} \chi_\ell \equiv 1$ ,  $\chi_\ell = \chi_1(2^\ell \cdot)$  and  $\chi_1$  is supported in  $(2, 8)$ . Let

$$\mathfrak{B}_{\lambda, \ell}[f, g] = \iint_{[-1, 1]^2} e^{is(|y|^2 + |z|^2) - i\frac{s}{\lambda^2} \xi(y+z)} \chi_\ell(|y-z|) f(y) g(z) dy dz,$$

so that

$$(\mathcal{E}f\mathcal{E}f)\left(\frac{s}{\lambda^2}\xi, s\right) = \sum_{\ell \geq 0} \mathfrak{B}_{\lambda, \ell}(f, f)(\xi, s).$$

We shall verify that for  $\ell \geq 0$

$$(4.2) \quad \|\mathfrak{B}_{\lambda, \ell}(f, g)\|_{L^{q/2}(\mathcal{A}(\lambda^2); L^{r/2}[\lambda^2, 2\lambda^2])} \lesssim 2^{-2\ell(\frac{1}{2} - \frac{1}{q} - \frac{1}{r})} \|f\|_p \|g\|_p$$

when  $(\frac{1}{p}, \frac{1}{q}, \frac{1}{r})$  is contained in the closed tetrahedron with vertices  $(0, 0, 0)$ ,  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ ,  $(\frac{1}{2}, 0, \frac{1}{2})$  and  $(0, 0, \frac{1}{2})$ . By summing a geometric series, this yields (2.4) for  $(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}) \in \Delta_1$ , which by Lemmata 2.2 and 2.4, yields the desired

$$(4.3) \quad \mathfrak{A}_\lambda(p; q, r) \lesssim \lambda^{1 - \frac{1}{p} - \frac{1}{q} - \frac{2}{r}}.$$

We remark that conversely, if (4.3) holds, then we can use Lemma 2.3 and a Fourier expansion of  $\chi_\ell(y-z)$  to bound the left hand side of (4.2) by  $C\|f\|_p\|g\|_p$ , with  $C$  independent of  $\ell$ .

It remains to show (4.2). By interpolation it is enough to do this with  $(p, q, r) = (\infty, \infty, \infty)$ ,  $(4, 4, 4)$ ,  $(2, \infty, 2)$ , and  $(\infty, \infty, 2)$ . The last two estimates were already obtained; note that the bounds (4.1) and (4.3) coincide for the cases  $(p, q, r) = (2, \infty, 2)$  and  $(\infty, \infty, 2)$  and the bounds for (4.2) are independent of  $\ell$ . Hence from the bounds (4.1) previously obtained and the discussion above we have the required bounds for  $(p, q, r) = (2, \infty, 2)$ , and  $(\infty, \infty, 2)$ . We note that the argument for the proof of the endpoint adjoint restriction theorem in [12] gives

$$(4.4) \quad \|B_{\lambda, \ell}(f, g)\|_{L_{\xi, s}^2} \lesssim \|f\|_4 \|g\|_4,$$

uniformly in  $\ell \geq 0$ , where  $B_{\lambda, \ell}(f, g)(\xi, s) = \mathfrak{B}(f, g)(\frac{\lambda^2}{s}\xi, s)$ , and by a change of variables we obtain (4.2) holds with  $(p, q, r) = (4, 4, 4)$ . To get the inequality (4.2) for  $(p, q, r) = (\infty, \infty, \infty)$  we need to integrate  $\chi_\ell(|y-z|)$  over  $[-1, 1]^2$  which yields the gain of  $2^{-\ell}$ .

4.2. *Proof of (ii).* We also consider the cases  $1 - \frac{1}{p} > \frac{2}{q} + \frac{1}{r}$  and  $1 - \frac{1}{p} \leq \frac{2}{q} + \frac{1}{r}$  separately.

*The case  $1 - \frac{1}{p} \leq \frac{2}{q} + \frac{1}{r}$ .* We note that the set

$$\Delta_2 = \left\{ \left( \frac{1}{p}, \frac{1}{q}, \frac{1}{r} \right) : 2 \leq p < r \leq q \leq \infty, \quad \frac{2}{q} + \frac{1}{r} \geq 1 - \frac{1}{p} \right\}$$

is the closed tetrahedron with vertices  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ ,  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ ,  $(\frac{1}{2}, \frac{1}{6}, \frac{1}{6})$  and  $(\frac{1}{2}, 0, \frac{1}{2})$ , from which the face with vertices  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ ,  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$  and  $(\frac{1}{2}, 0, \frac{1}{2})$  is removed. Note that from the previous bounds (4.1) and (4.3) we already have the required bounds

$$(4.5) \quad \mathfrak{A}_\lambda(p; q, r) \lesssim \lambda^{\frac{1}{q} - \frac{1}{r}}$$

for  $(p, q, r) = (2, 2, 2)$  and  $(2, \infty, 2)$ . Obviously  $\Delta_2$  is contained in the convex hull of  $(\frac{1}{2}, 0, \frac{1}{2})$ ,  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ , and the half open line segment  $[(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}), (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})]$ . Hence by it is enough to show (4.5) for  $(\frac{1}{p}, \frac{1}{q}, \frac{1}{r})$  contained in the half closed line segment  $[(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}), (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})]$ . But these follow from Lemmata 2.2 and 2.4, combined with the restriction estimate for the parabola which gives (2.4) for  $(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}) \in [(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}), (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})]$ .

*The case  $1 - \frac{1}{p} > \frac{2}{q} + \frac{1}{r}$ .* We note that the set

$$\left\{ \left( \frac{1}{p}, \frac{1}{q}, \frac{1}{r} \right) : 2 \leq p < r \leq q \leq \infty, \quad \frac{2}{q} + \frac{1}{r} < 1 - \frac{1}{p} \right\}$$

is contained in the quadrangular pyramid  $\mathcal{Q}$  with vertices  $(0, 0, 0)$ ,  $(\frac{1}{2}, 0, 0)$ ,  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ ,  $(\frac{1}{2}, \frac{1}{6}, \frac{1}{6})$ , and  $(\frac{1}{2}, 0, \frac{1}{2})$ . We need to show (4.3) for  $(\frac{1}{p}, \frac{1}{q}, \frac{1}{r})$  contained in the above set. Repeating the above argument, the asserted estimates follow if we establish, for  $\ell \geq 0$  and  $(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}) \in \mathcal{Q}$ ,

$$(4.6) \quad \|\mathfrak{B}_{\lambda, \ell}(f, g)\|_{L^{q/2}(\mathcal{A}(\lambda^2); L^{r/2}[\lambda^2, 2\lambda^2])} \lesssim 2^{-\ell(1 - \frac{1}{p} - \frac{2}{q} - \frac{1}{r})} \|f\|_p \|g\|_p.$$

We only need to verify it for  $(p, q, r) = (\infty, \infty, \infty)$ ,  $(4, 4, 4)$ ,  $(2, \infty, 2)$ ,  $(2, 6, 6)$ , and  $(2, \infty, \infty)$ . The first three cases were already obtained when we showed (4.2), and the case  $(p, q, r) = (2, 6, 6)$  follows from the linear adjoint restriction estimate for the parabola as before. Finally the case  $(p, q, r) = (2, \infty, \infty)$  with a gain of  $2^{-\ell/2}$  follows from the Schwarz inequality, and so we are done.

## 5. SHARPER REGULARITY RESULTS

5.1. *Combining frequency localized pieces.* One can use the uniform regularity results for the frequency localized pieces to prove sharper bounds such as Sobolev estimates by using arguments based on the Fefferman–Stein  $\#$ -function. Let  $\varphi$  be a radial smooth function supported in  $\{\xi : 1/4 < |\xi| < 4\}$ , not identically 0. Let  $I = [-1, 1]$  and

$$(5.1) \quad \Gamma(p; q, r) = \sup_{\lambda > 1} \lambda^{-d(-\frac{1}{p} - \frac{1}{q}) + \frac{2}{r}} \left\| U\varphi\left(\frac{D}{\lambda}\right) \right\|_{L^p \rightarrow L^q(\mathbb{R}^d; L^r(I))}$$

It is not hard to verify that the finiteness of  $\Gamma(p; q, r)$  is independent of the particular choice of  $\varphi$ . The following statement is a special case of the result in the appendix of [16].

**Proposition 5.1.** *Let  $p_0, q_0, r_0 \in [1, \infty]$ ,  $q \in (q_0, \infty)$ ,  $r_0 \leq r < \infty$ ,  $p_0 \leq q_0$  and assume  $1/p_0 - 1/q_0 = 1/p - 1/q$ . Suppose that  $\Gamma(p_0; q_0, r_0) < \infty$ . Then*

$$(5.2) \quad \left\| \left( \int_I |Uf(\cdot, t)|^r dt \right)^{1/r} \right\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{B_{\alpha, q}^p(\mathbb{R}^d)}, \quad \alpha = d\left(1 - \frac{1}{p} - \frac{1}{q}\right) - \frac{2}{r}.$$

If  $f \in B_{s, q}^p(\mathbb{R}^d)$  with  $s = d\left(1 - \frac{1}{p} - \frac{1}{q}\right)$ , then for almost every  $x \in \mathbb{R}^d$  the function  $t \mapsto U^\alpha f(x, t)$  is locally in  $B_{1/q, \nu}^q(\mathbb{R})$ , and (thus) continuous, and

$$\left\| \sup_{t \in I} |Uf(\cdot, t)| \right\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{B_{s, q}^p(\mathbb{R}^d)}, \quad s = d\left(1 - \frac{1}{p} - \frac{1}{q}\right).$$

The Sobolev estimates follow from this since for  $q \geq p \geq 2$  one has  $L_\alpha^p \subset B_{\alpha, p}^p \subset B_{\alpha, q}^p$ . We note that the result in [16] is slightly sharper. Namely the left hand side of (5.2) can be replaced by the  $L^q(\mathbb{R}^d)$  norm of  $(\sum_{k>0} (\int_I |P_k Uf(\cdot, t)|^r dt)^{\nu/r})^{1/\nu}$ , where  $\nu > 0$ .

*Proof of Corollaries 1.4 and 1.7.* Proposition 5.1 implies the validity of the corollaries given their analogues for frequency localized functions (namely Theorems 1.3 and 1.6). For Corollary 1.7 we use that  $R^*(p_0 \rightarrow q_0)$  implies  $R^*(p \rightarrow q_0)$  for all  $p \geq p_0$ .  $\square$

5.2. *A remark on recent results by Bourgain and Guth.* As mentioned in the introduction, the recent results in [3] on  $R^*(q \rightarrow q)$  give results on the sharp  $L_\alpha^q \rightarrow L^q(\mathbb{R}^d \times I)$  boundedness of  $U$ . In a restricted range they also imply new results on  $R^*(p \rightarrow q)$  with the best possible  $q = q(p)$  which Tao [28] had proved for  $q > \frac{2(d+3)}{d+1}$ , and likewise one then obtains corresponding results for the Schrödinger operator. The following statement is proved by a simple interpolation argument for bilinear operators.

**Proposition 5.2.** *Suppose that  $R^*(q_0 \rightarrow q_0)$  holds for some  $q_0 \in (2, \frac{2(d+3)}{d+1})$ . Then*

(i)  $R^*(p \rightarrow q)$  holds with  $q = \frac{d+2}{d} p'$  provided that

$$q > q_* := \frac{2(d+3)}{d+1} \left(1 - \gamma(d, q_0)\right), \quad \text{where } \gamma(d, q_0) = \frac{\frac{1}{q_0} - \frac{d+1}{2(d+3)}}{\frac{d+1}{2d} - \frac{d+2}{dq_0}}.$$

(ii) Let  $q_* < q < \infty$ ,  $q \leq r \leq \infty$  and suppose that  $0 \leq \frac{1}{p} - \frac{1}{q} < 1 - \frac{2(d+1)}{dq_*}$ . Then  $U : L_\alpha^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d; L^r(I))$  is bounded with  $\alpha = d\left(1 - \frac{1}{p} - \frac{1}{q}\right) - \frac{2}{r}$ .

In two dimensions  $R^*(q \rightarrow q)$  was proven in [3] for  $q > 33/10$  and the sharp inequality  $R^*(p \rightarrow q)$  for  $q = 2p'$  follows for  $q > 63/19$ .

*Proof of Proposition 5.2.* By Theorem 1.6 and Proposition 5.1 it suffices to prove the first part.

Let  $E_1$  and  $E_2$  be  $1/2$ -separated sets in the unit ball of  $\mathbb{R}^d$  and define  $\mathcal{E}_i f = \mathcal{E}[f\chi_{E_i}]$ . By Theorem 2.2 in [29], it suffices to prove the estimate

$$(5.3) \quad \|\mathcal{E}_1 f_1 \mathcal{E}_2 f_2\|_{q/2} \lesssim \|f_1\|_p \|f_2\|_p$$

for  $q > q_*$  and  $p$  in a neighbourhood of  $\frac{dq}{dq-d-2}$  (i.e. the  $p$  which satisfies  $q = \frac{d+2}{d}p'$ ).

By hypothesis and Hölder's inequality, (5.3) holds with  $p \geq q = q_0$ . By Tao's theorem (5.3) holds with  $p \geq 2$  and  $q/2 > \frac{d+3}{d+1}$ . The theorem then follows by interpolation of bilinear operators. Indeed, we determine  $\theta \in (0, 1)$  and  $q_* \in (q_0, \frac{2(d+3)}{d+1})$  by

$$\frac{1-\theta}{2} + \frac{\theta}{q_0} = 1 - \frac{d+2}{dq_*}, \quad (1-\theta)\frac{d+1}{d+3} + \theta\frac{2}{q_0} = \frac{2}{q_*}.$$

We compute  $\theta = (\frac{d+2}{dq_*} - \frac{1}{2}) / (\frac{1}{2} - \frac{1}{q_0})$  and  $\theta = (\frac{1}{q_*} - \frac{d+1}{2(d+3)}) / (\frac{1}{q_*} - \frac{d+1}{2(d+3)})$ , from which we obtain  $1/q_* = (\frac{d+1}{2(d+3)} - \frac{b}{2}) / (1 - \frac{d+2}{d}b)$  with  $b = (\frac{1}{q_0} - \frac{d+1}{2(d+3)}) / (\frac{1}{2} - \frac{1}{q_0})$ . A further computation shows that  $q_*$  is equal to  $\frac{2(d+3)}{d+1}(1 - \gamma(d, q_0))$  as in the statement of the lemma.  $\square$

## 6. PROOF OF THEOREM 1.8

**Definition.** Fix  $d \geq 1$ , and let  $p, q, r \in [2, \infty]$ . For  $N > 1$ , let

$$\Lambda_{p,q,r}(N, \rho) \equiv \Lambda_{p,q,r}(N, \rho; d) = \sup \|Uf_1 Uf_2\|_{L^{q/2}(\mathbb{R}^d, L^{r/2}[0, \rho])}$$

where the supremum is taken over all pairs of function  $(f_1, f_2)$  whose Fourier transforms are supported in  $1$ -separated subsets of  $\{\xi : |\xi - Ne_1| \leq 2d\}$ , and which satisfy  $\|f_1\|_p, \|f_2\|_p \leq 1$ .

We remark that the unit vector  $e_1$  does not play a special role here. It could be replaced by any unit vector, by rotational invariance.

By considering two bump functions, it is easy to calculate that

$$(6.1) \quad \Lambda_{p,q,r}(N, \rho) \gtrsim N^{\frac{2}{q} - \frac{2}{r}}, \quad 1 \leq p, q, r \leq \infty,$$

whenever  $\rho > 1$ , and significant for Theorem 1.8 is the following two dimensional estimate,

$$(6.2) \quad \sup_{\rho > 1} \Lambda_{2,q,r}(N, \rho; 2) \lesssim N^{\frac{2}{q} - \frac{2}{r}}, \quad q > 16/5, \quad r \geq 4,$$

which was proven in [17] (see also [15] and [22] for related previous results). We will combine this with the following two lemmata.

**Lemma 6.1.** *Let  $p_0 \leq p \leq q \leq r$  and  $\varepsilon_0 > 0$ . Then, for  $N, \rho > 1$ ,*

$$(6.3) \quad \Lambda_{p,q,r}(N, \rho) \lesssim N^{\varepsilon_0} \rho^{2d(\frac{1}{p_0} - \frac{1}{p})} \Lambda_{p_0,q,r}(N, \rho).$$

**Lemma 6.2.** *Let  $2 \leq p \leq q \leq r \leq \frac{2q}{q-2}$  and  $\varepsilon > 0$ . Let  $\psi \in C_c^\infty$  be supported in the annulus  $\{\xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 2\}$ . Then, for  $\lambda > 1$ ,*

$$(6.4) \quad \begin{aligned} & \|U\psi(\frac{D}{\lambda})f\|_{L^q(\mathbb{R}^d, L^r[0,1])} \\ & \lesssim \left( \lambda^{\frac{4}{q} - 2d(\frac{1}{p} - \frac{1}{q})} + \sup_{1 < N < \lambda} N^{\frac{4}{r} - 2d(\frac{1}{p} - \frac{1}{q}) + \varepsilon} \Lambda_{p,q,r}(N, C\lambda^2/N^2) \right)^{1/2} \lambda^{-\frac{2}{r} + d(\frac{1}{p} - \frac{1}{q})} \|f\|_p. \end{aligned}$$

Lemma 6.1 relies on a localization argument such as in [14] and Lemma 6.2 relies on a by now standard scaling argument in [29] which reduces estimates for bilinear operators with separation assumptions to estimates for linear operators.

We may combine (6.3), with  $p_0 = 2$ , and (6.4) to obtain

**Corollary 6.3.** *Let  $2 \leq p \leq q \leq r \leq \frac{2q}{q-2}$ . Suppose that*

$$(6.5) \quad \sup_{\rho > 1} \Lambda_{2,q,r}(N, \rho; d) \lesssim N^\gamma, \quad \text{for some } \gamma < 2d\left(1 - \frac{1}{p} - \frac{1}{q}\right) - \frac{4}{r}.$$

*Then if  $d(1 - \frac{1}{p} - \frac{1}{q}) - \frac{2}{q} \geq 0$ , then for all  $\lambda > 1$ ,*

$$(6.6) \quad \left\| U\psi\left(\frac{D}{\lambda}\right)f \right\|_{L^q(\mathbb{R}^d; L^r[0,1])} \lesssim \lambda^{d(1-\frac{1}{p}-\frac{1}{q})-\frac{2}{r}} \|f\|_p.$$

Supposing this for the moment we give the

*Proof of Theorem 1.8.* By Proposition 5.1 it suffices to prove, in two spatial dimensions, the estimate (6.6) for  $p = q > 16/5$  and  $r \geq 4$ . Using (6.2), we put  $\gamma = 2/q - 2/r$  and verify that the condition (6.5) with  $d = 2$  in the range  $p = q > 16/5$  and  $r \geq 4$ . Thus (6.6) holds in this range, and we are done.  $\square$

*Proof of Lemma 6.1.* Let  $\eta_1, \eta_2$  be smooth, supported in balls of diameter  $1/2$  which are contained in  $\{\xi : |\xi - Ne_1| \leq 2d\}$ , and which are separated by  $1/2$ . Define the operators  $S_1, S_2$  by  $\widehat{S_i f}(\xi, t) = \eta_i(\xi) \widehat{Uf}(\xi)$ ,  $i = 1, 2$ . It suffices to prove that  $\|S_1 f_1 S_2 f_2\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0, \rho])}$  is dominated by  $\|f_1\|_p \|f_2\|_p$  times a constant multiple of the expression on the right hand side of (6.3).

We partition  $\mathbb{R}^d$  into cubes  $\mathcal{Q}_\nu$  of side  $\rho$  with centre  $\rho\nu \in \rho\mathbb{Z}^d$ , and define

$$(6.7) \quad \mathcal{P}_\nu = \{(x, t) \in \mathbb{R}^d \times [0, \rho] : x - 2tNe_1 \in \mathcal{Q}_\nu\}.$$

The parallelepipeds form a partition of  $\mathbb{R}^d \times [0, \rho]$ . For fixed  $x$  the intervals  $I_\nu^x = \{t : (x, t) \in \mathcal{P}_\nu\}$  are disjoint. Thus

$$\|F\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0, \rho])}^{q/2} \leq \int_{\mathbb{R}^d} \left( \sum_\nu \int_{I_\nu^x} |F(x, t)|^{r/2} dt \right)^{q/r} dx \leq \sum_\nu \|\chi_{\mathcal{P}_\nu} F\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0, \rho])}^{q/2};$$

here we used the triangle inequality for  $\|\cdot\|_{\rho^{q/r}}$  as  $q/r \leq 1$ .

Taking  $F = S_1 f_1 S_2 f_2$ , and denoting by  $\mathcal{Q}_\nu^*$ , the enlarged cube with side  $50d\rho N^\varepsilon$ , where  $0 < \varepsilon < 4d\varepsilon_0$ , we obtain

$$\begin{aligned} \|S_1 f_1 S_2 f_2\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0, \rho])}^{q/2} &\leq \sum_\nu \|\chi_{\mathcal{P}_\nu} S_1 f_1 S_2 f_2\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0, \rho])}^{q/2} \\ &\lesssim \sum_\nu (I_\nu^{q/2} + II_\nu^{q/2} + III_\nu^{q/2} + IV_\nu^{q/2}), \end{aligned}$$

where

$$(6.8) \quad \begin{aligned} I_\nu &= \|\chi_{\mathcal{P}_\nu} S_1 [f_1 \chi_{\mathcal{Q}_\nu^*}] S_2 [f_2 \chi_{\mathcal{Q}_\nu^*}]\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0, \rho])}, \\ II_\nu &= \|\chi_{\mathcal{P}_\nu} S_1 [f_1 \chi_{\mathbb{R}^d \setminus \mathcal{Q}_\nu^*}] S_2 [f_2 \chi_{\mathcal{Q}_\nu^*}]\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0, \rho])}, \\ III_\nu &= \|\chi_{\mathcal{P}_\nu} S_1 [f_1 \chi_{\mathcal{Q}_\nu^*}] S_2 [f_2 \chi_{\mathbb{R}^d \setminus \mathcal{Q}_\nu^*}]\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0, \rho])}, \\ IV_\nu &= \|\chi_{\mathcal{P}_\nu} S_1 [f_1 \chi_{\mathbb{R}^d \setminus \mathcal{Q}_\nu^*}] S_2 [f_2 \chi_{\mathbb{R}^d \setminus \mathcal{Q}_\nu^*}]\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0, \rho])}. \end{aligned}$$

First we consider the main terms  $I_\nu$ . By Hölder's inequality,

$$I_\nu \leq \Lambda_{p_0, q, r}(N, \rho) \prod_{i=1}^2 \|f_i \chi_{\mathcal{Q}_\nu^*}\|_{p_0} \lesssim \Lambda_{p_0, q, r}(N, \rho) (\rho N^\varepsilon)^{2d(\frac{1}{p_0} - \frac{1}{p})} \prod_{i=1}^2 \|f_i \chi_{\mathcal{Q}_\nu^*}\|_p$$

We use the Schwarz inequality, the embedding  $\ell^p \subset \ell^q$ ,  $p \leq q$ , and the fact that every  $x$  is contained in only  $O(N^{\varepsilon d})$  of the cubes  $\mathcal{Q}_\nu^*$  to get

$$\sum_\nu \prod_{i=1}^2 \|f_i \chi_{\mathcal{Q}_\nu^*}\|_p^{q/2} \leq \prod_{i=1}^2 \left( \sum_\nu \|f_i \chi_{\mathcal{Q}_\nu^*}\|_p^q \right)^{1/2} \lesssim N^{\varepsilon d} \prod_{i=1}^2 \|f_i\|_p^{q/2}.$$

Combining the previous two estimates we bound

$$(6.9) \quad \left( \sum_\nu I_\nu^{q/2} \right)^{2/q} \lesssim N^{2d\varepsilon(\frac{1}{p_0} - \frac{1}{p} + \frac{1}{q})} \rho^{2d(\frac{1}{p_0} - \frac{1}{p})} \Lambda_{p_0, q, r}(N, \rho) \prod_{i=1}^2 \|f_i\|_p.$$

We use very crude estimates to handle the remaining three terms which can to be dominated by  $C_{M, \varepsilon} (N^\varepsilon \rho)^{-M} \|f_1\|_p \|f_2\|_p$ , which finishes the proof since  $\Lambda_{p_0, q, r}(N, \rho) \gtrsim N^{\frac{2}{q} - \frac{2}{r}}$  by (6.1).

We only give the argument to bound  $\sum_\nu II_\nu^{q/2}$  as the other terms are handled similarly. By the Schwarz inequality we estimate  $\sum_\nu II_\nu^{q/2}$  by

$$(6.10) \quad \left( \sum_\nu \|\chi_{\mathcal{P}_\nu} S_1 [f_1 \chi_{\mathbb{R}^d \setminus \mathcal{Q}_\nu^*}]\|_{L^q(\mathbb{R}^d; L^r[0, \rho])}^q \right)^{1/2} \left( \sum_\nu \|S_2 [f_2 \chi_{\mathcal{Q}_\nu^*}]\|_{L^q(\mathbb{R}^d; L^r[0, \rho])}^q \right)^{1/2}.$$

For the second factor we use a wasteful bound, namely that the  $L^p \rightarrow L^q(\mathbb{R}^d; L^r[0, \rho])$  operator norm of  $S_2$  is  $O(\rho^{1/r} N^d)$ . Consequently, the second factor in (6.10) can be bounded by  $C \rho^{q/(2r)} N^{d(\varepsilon + q/2)} \|f_2\|_p^{q/2}$ .

We consider the first factor in (6.10) and write  $S_1 f(x, t) = \mathcal{K}_t * f(x)$  where

$$\mathcal{K}_t(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \chi(\xi - Ne_1) e^{-it|\xi|^2 + i\langle y, \xi \rangle} d\xi$$

with  $\chi \in C_c^\infty$  equal to 1 on the ball of radius  $2d$  centred at the origin. Integration by parts yields that for every  $t \in [0, \rho]$

$$|\mathcal{K}_t(y)| \leq C_M |y - 2tNe_1|^{-M} \quad \text{if } |y - 2tNe_1| \geq 4d\rho.$$

Let  $\mathbf{c}_\nu$  be the centre of  $\mathcal{Q}_\nu^*$ . If  $x - y \in \mathbb{R}^d \setminus \mathcal{Q}_\nu^*$  and  $(x, t) \in \mathcal{P}_\nu$ , then  $|x - y - \mathbf{c}_\nu| \geq 10d\rho N^\varepsilon$ ,  $|x - 2tNe_1 - \mathbf{c}_\nu| \leq 2d\rho N^\varepsilon$ , and therefore also  $|y - 2tNe_1| \geq 8d\rho N^\varepsilon$ . Thus for this choice of  $(x, t)$  and  $y$  we have

$$|S_1 [f_1 \chi_{\mathbb{R}^d \setminus \mathcal{Q}_\nu^*}]| \lesssim (\rho N^\varepsilon)^{-M+d+1} \int_{|y - 2tNe_1| \geq 8d\rho N^\varepsilon} \frac{|f_1(x - y)|}{|y - 2tNe_1|^{d+1}} dy$$

and the integral is bounded by  $(\rho N)^{d+1} \int (1 + |y|)^{-d-1} |f_1(x - y)| dy$ . Here we use  $\rho > 1$ . Now let  $\mathcal{Q}_\nu^{**}$  be the cube of sidelength  $\rho(2 + N)$  centred at  $\mathbf{c}_\nu$ ; then  $\mathcal{Q}_\nu^{**} \times [0, \rho]$  contains  $\mathcal{P}_\nu$ . Letting  $\mathcal{C}_{\rho, N} := \rho^{1/r} (\rho N^\varepsilon)^{-M_1 + d + 1} (\rho N)^{d+1}$ , we have

$$\sum_\nu \|\chi_{\mathcal{P}_\nu} S_1 [f_1 \chi_{\mathbb{R}^d \setminus \mathcal{Q}_\nu^*}]\|_{L^q(\mathbb{R}^d; L^r[0, \rho])}^q \lesssim \mathcal{C}_{\rho, N}^q \sum_\nu \int_{\mathcal{Q}_\nu^{**}} \left| \int \frac{|f_1(x - y)|}{(1 + |y|)^{d+1}} dy \right|^q dx$$

which is  $\lesssim \mathcal{C}_{\rho, N}^q (\rho N)^{(d+1)} \|f_1\|_p^q$ ; here one uses Young's inequality and the fact that each  $x \in \mathbb{R}^d$  is contained in at most  $O((\rho N)^{d+1})$  of the cubes  $\mathcal{Q}_\nu^{**}$ . Collecting the estimates yields the crude bound

$$\sum_\nu II_\nu^{q/2} \leq C_M (\rho N^\varepsilon)^{-M} (\rho N)^{10dq} \|f_1\|_p^{q/2} \|f_2\|_p^{q/2},$$

and we conclude by choosing  $M$  sufficiently large.  $\square$

*Proof of Lemma 6.2.* For  $j \geq 0$ , we write

$$A(j, \lambda) := 2^{2j(\frac{2}{r}-d(\frac{1}{p}-\frac{1}{q}))} \sup_{2^{j-1} \leq N \leq 2^{j+1}} \Lambda_{p,q,r}(N, C\lambda^2 2^{-2j+1}).$$

Define  $T = U\psi(D)$ , and thus  $U\psi(\frac{D}{\lambda})f(x, t) = T[f(\lambda^{-1}\cdot)](\lambda x, \lambda^2 t)$ . By scaling,

$$(6.11) \quad \|U\psi(\frac{D}{\lambda})\|_{L^p \rightarrow L^q(\mathbb{R}^d; L^r[0,1])} = \lambda^{-\frac{2}{r}+d(\frac{1}{p}-\frac{1}{q})} \|T\|_{L^p \rightarrow L^q(\mathbb{R}^d; L^r[0,\lambda^2])},$$

so that the statement of the lemma is an immediate consequence of

$$(6.12) \quad \|T\|_{L^p \rightarrow L^q(\mathbb{R}^d; L^r[0,\lambda^2])} \lesssim \left( \lambda^{\frac{4}{q}-2d(\frac{1}{p}-\frac{1}{q})} + \sum_{1 \leq 2^j \leq \lambda} A(j, \lambda) \right)^{1/2}.$$

Now by scaling we have that

$$(6.13) \quad \|Tf_1 Tf_2\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0,\lambda^2])} \lesssim A(j, \lambda) \prod_{i=1}^2 \|f_i\|_p,$$

whenever  $\widehat{f}_1$  and  $\widehat{f}_2$  are supported in a  $2^{-j+1}$  ball, contained in  $\{\xi : 1/2 < |\xi| \leq 2\}$ , and their supports are  $2^{-j}$ -separated. We will also require the following simpler estimates

$$(6.14) \quad \|Tf_1 Tf_2\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0,\lambda^2])} \lesssim \lambda^{\frac{4}{q}-2d(\frac{1}{p}-\frac{1}{q})} \prod_{i=1}^2 \|f_i\|_p,$$

whenever  $\widehat{f}_1$  and  $\widehat{f}_2$  are supported in an ball of radius  $\lambda^{-1}$ , contained in  $\{\xi : 1/2 < |\xi| \leq 2\}$ .

By the Schwarz inequality, this follows from  $\|Tf_1\|_{L^q(\mathbb{R}^d; L^r[0,\lambda^2])} \lesssim \lambda^{\frac{2}{q}-d(\frac{1}{p}-\frac{1}{q})} \|f_1\|_p$ . Let  $t \mapsto \varpi(t)$  be a Schwartz function which is positive on  $[0, 4d]$  and whose Fourier transform is supported in  $[-1, 1]$ . By scaling and rotation this would follow from

$$(6.15) \quad \|\varpi Tf\|_{L^q(\mathbb{R}^d; L^r(\mathbb{R}))} \lesssim \lambda^{\frac{2}{q}-\frac{2}{r}} \|f\|_p$$

whenever  $\widehat{f}$  is supported in  $\{\xi : |\xi - \lambda e_1| \leq 2d\}$ . By a change of variables and trivial estimates it is easy to see (6.15) for  $1 \leq p \leq q = r \leq \infty$ . The estimate for  $r > q$  follows by applying Bernstein's inequality in  $t$  since the temporal Fourier transform of  $\varpi Tf$  is contained in  $\{s : s \sim \lambda^2\}$ .

We now argue similarly as in [29]. Write  $\|Tf\|_{L^q(\mathbb{R}^d; L^r[0,\lambda^2])}^2 = \|Tf Tf\|_{L^{q/2}(\mathbb{R}^d; L^{r/2}[0,\lambda^2])}$ . For each  $j$ ,  $1 \leq 2^j \leq 2\lambda$ , we tile  $\mathbb{R}^d$  with dyadic cubes  $s_\ell^j = \prod_{i=1}^d [2^{-j}\ell_i, 2^{-j}\ell_{i+1})$  of sidelength  $2^{-j}$ , indexed by  $\ell \in \mathbb{Z}^d$ . For  $j$ ,  $1 \leq 2^j \leq \lambda$ , we write  $\ell \sim_j \tilde{\ell}$  if  $s_\ell^j$  and  $s_{\tilde{\ell}}^j$  have adjacent parents, but are not adjacent. When  $\lambda < 2^j \leq 2\lambda$ , we mean by  $\ell \sim_j \tilde{\ell}$  that the distance between  $s_\ell^j$  and  $s_{\tilde{\ell}}^j$  is  $\lesssim \lambda^{-1}$ . Then, we then can write for every  $(\xi, \eta) \in \mathbb{R}^d$ , with  $\xi \neq \eta$ ,

$$(6.16) \quad \sum_{1 \leq 2^j \leq 2\lambda} \sum_{\substack{(\ell, \tilde{\ell}) \\ \ell \sim_j \tilde{\ell}}} \chi_{s_\ell^j}(\xi) \chi_{s_{\tilde{\ell}}^j}(\eta) = 1$$

Define  $P_\ell^j$  by  $\widehat{P_\ell^j f} = \chi_{s_\ell^j} \widehat{f}$ ; then the operators  $P_\ell^j$  are bounded on  $L^p$ ,  $1 < p < \infty$ , with operator norms independent of  $\ell$  and  $j$ . For any Schwartz function  $f$  we have by (6.16)

$$[Tf(x, t)]^2 = \sum_{1 \leq 2^j \leq 2\lambda} \sum_{(\ell, \tilde{\ell}) : \ell \sim_j \tilde{\ell}} TP_\ell^j f(x, t) TP_{\tilde{\ell}}^j f(x, t)$$

Let  $\varphi \in C_c^\infty$  be supported in  $[-1, 1]^d$ , satisfying  $\sum_{\mathfrak{z} \in \mathbb{Z}^d} \varphi(\xi - \mathfrak{z}) = 1$  for all  $\xi \in \mathbb{R}^d$ . Define  $\tilde{P}_3^j$  as acting on  $L^a(L^b)$  functions by  $\tilde{P}_3^j \widehat{G}(\xi, t) = \varphi(2^j \xi - \mathfrak{z}) \widehat{G}(\xi, t)$ . We use the inequality

$$(6.17) \quad \left\| \sum_{\mathfrak{z}} \tilde{P}_3^j G_{\mathfrak{z}} \right\|_{L^a(L^b)} \leq C \|\{G_{\mathfrak{z}}\}\|_{\ell^a(L^a(L^b))}, \quad 1 \leq a \leq 2, \quad a \leq b \leq a',$$

The constant  $C$  in (6.17) is independent of  $j$ . The inequality follows from Plancherel's theorem in the case  $a = b = 2$ , and from an application of Minkowski's inequality in the case  $a = 1, 1 \leq b \leq \infty$ . The intermediate cases follow by interpolation. Note that for any  $j$  and any  $\mathfrak{z} \in \mathfrak{Z}^d$  the number of pairs  $(\ell, \tilde{\ell})$  with  $\ell \sim_j \tilde{\ell}$  for which  $\tilde{P}_3^j [TP_\ell^j f TP_{\tilde{\ell}}^j f] \neq 0$  is uniformly bounded (independent of  $j, \mathfrak{z}, f$ ). Thus inequality (6.17) applied with  $a = q/2, b = r/2$ , implies

$$(6.18) \quad \|Tf\|_{L^q(L^r[0, \lambda^2])}^2 \lesssim \sum_{1 \leq 2^j \leq 2\lambda} \left( \sum_{\ell \sim_j \tilde{\ell}} \|TP_\ell^j f TP_{\tilde{\ell}}^j f\|_{L^{q/2}(L^{r/2}[0, \lambda^2])}^{q/2} \right)^{2/q};$$

here we use that  $1 \leq q/2 \leq r/2 \leq (q/2)'$  (i.e.  $q \leq r \leq \frac{2q}{q-2}$  which implies that  $q/2 \leq 2$ .)

Now by (6.13) and (6.14) the right hand side of (6.18) is dominated by a constant times

$$\begin{aligned} \sum_{1 \leq 2^j \leq \lambda} A(j, \lambda) \left( \sum_{\ell \sim_j \tilde{\ell}} \|P_\ell^j f\|_p^{q/2} \|P_{\tilde{\ell}}^j f\|_p^{q/2} \right)^{2/q} + \lambda^{\frac{4}{q} - 2d(\frac{1}{p} - \frac{1}{q})} \left( \sum_{\ell \sim_{j_0} \tilde{\ell}} \|P_\ell^{j_0} f\|_p^{q/2} \|P_{\tilde{\ell}}^{j_0} f\|_p^{q/2} \right)^{2/q} \\ \lesssim \lambda^{\frac{4}{q} - 2d(\frac{1}{p} - \frac{1}{q})} \left( \sum_{\ell} \|P_\ell^{j_0} f\|_p^q \right)^{2/q} + \sum_{1 \leq 2^j \leq \lambda} A(j, \lambda) \left( \sum_{\ell} \|P_\ell^j f\|_p^q \right)^{2/q}. \end{aligned}$$

Here  $j_0$  is the integer such that  $\lambda < 2^{j_0} \leq 2\lambda$ , and we have used the Schwarz inequality and the fact that for each  $(j, \ell)$  the number of  $\tilde{\ell}$  with  $\ell \sim_j \tilde{\ell}$  is uniformly bounded. Since  $2 \leq p \leq q$ , we also have

$$\left( \sum_{\ell} \|P_\ell^j f\|_p^q \right)^{1/q} \lesssim \|f\|_p,$$

and thus we have shown (6.12).  $\square$

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