

# LEBESGUE SPACE ESTIMATES FOR A CLASS OF FOURIER INTEGRAL OPERATORS ASSOCIATED WITH WAVE PROPAGATION

SANGHYUK LEE    ANDREAS SEEGER

ABSTRACT. We prove  $L^q$  estimates related to Sogge's conjecture for a class of Fourier integral operators associated with wave equations.

## 1. INTRODUCTION

In this note we prove a variable coefficient version of a recent result in [6] on the local  $L^q$  space-time regularity results for solutions of wave equations. The solution operators are Fourier integral operators satisfying the 'cinematic curvature' hypothesis introduced in [17] (see also [14]).

For the general setup let  $\mathcal{Y}$  and  $\mathcal{Z}$  be paracompact  $C^\infty$  manifolds,  $\dim(\mathcal{Y}) = d$ ,  $\dim(\mathcal{Z}) = d + 1$ ; in the current paper we shall need to assume  $d \geq 4$ . We are interested in sharp local regularity estimates for Fourier integral operators  $\mathcal{F} \in I^{\mu-1/4}(\mathcal{Z}, \mathcal{Y}; \mathcal{C})$  (associated with the Fourier integral distributions defined in [8]). Here the canonical relation

$$\mathcal{C} \subset T^*\mathcal{Z} \setminus 0_L \times T^*\mathcal{Y} \setminus 0_R$$

is a conic manifold of dimension  $2d + 1$ , which is Lagrangian with respect to the symplectic form  $d\zeta \wedge dz - d\eta \wedge dy$ . We denote by  $0_L$  and  $0_R$  the zero-sections in  $T^*\mathcal{Z}$  and  $T^*\mathcal{Y}$ , respectively.

We formulate a curvature hypothesis which appeared in [3], [10] for classes of oscillatory integral operators (see [13], [2] for current results on these classes). We follow the exposition in [14] and impose conditions on the following projection maps.

$$\begin{array}{ccc} & \mathcal{C} & \\ & \swarrow \downarrow \searrow & \\ T^*\mathcal{Y} \setminus 0 & \mathcal{Z} & T^*\mathcal{Z} \setminus 0 \end{array}$$

We require that the projection  $\pi_L : \mathcal{C} \rightarrow T^*\mathcal{Y}$  is a submersion (*i.e.* the differential has maximal rank  $2d$ ). We also require that the space projection  $\Pi_{\mathcal{Z}} : \mathcal{C} \rightarrow \mathcal{Z}$  is a submersion (*i.e.* its differential has maximal rank  $d + 1$ ). As discussed in §2 of [14] this implies that for fixed  $z \in \Pi_{\mathcal{Z}}\mathcal{C}$  the image of the projection to the fiber,

$$\Gamma_z = \{\zeta : \in T^*\mathcal{Z} : \exists(y, \eta) \text{ such that } (z, \zeta, y, \eta) \in \mathcal{C}\},$$

is an immersed conic  $(d - 1)$ -dimensional hypersurface in  $T^*\mathcal{Z} \setminus 0_L$ . We then make an assumption on the curvature of the cones  $\Gamma_z$ :

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*Date:* September 23, 2011.

*Key words and phrases.* Space-time estimates for wave equations on compact manifolds, cinematic curvature, Fourier integral operators.

Supported in part by NRF grant 2011-0001251 (Korea) and and NSF grant 0652890 (USA).

**Curvature hypothesis  $\mathcal{H}(\ell)$ :**  $\mathcal{C} \subset T^*\mathcal{Z} \setminus 0_L \times T^*\mathcal{Y} \setminus 0_R$ , the projections  $\pi_R$  and  $\Pi_{\mathcal{Z}}$  are submersions and for each  $z$  the cone  $\Gamma_z$  has at least  $\ell$  nonvanishing principal curvatures at any  $\zeta \in \Gamma_z$ .

**Theorem 1.1.** *Let  $\ell \geq 3$ , let  $\mathcal{C}$  satisfy hypothesis  $\mathcal{H}(\ell)$  and let  $\mathcal{F} \in I^{\mu-1/4}(\mathcal{Z}, \mathcal{Y}; \mathcal{C})$ . Suppose  $\frac{2\ell}{\ell-2} < q < \infty$  and  $\mu \leq \frac{d}{q} - \frac{d-1}{2}$ . Then  $\mathcal{F}$  maps  $L_{\text{comp}}^q(\mathcal{Y})$  to  $L_{\text{loc}}^q(\mathcal{Z})$ .*

We may apply the theorem with  $\ell = d - 1$  to solutions of the wave equation on a compact Riemannian manifold  $M$ , with initial data in  $L^q$ -Sobolev spaces  $L_{\alpha}^q(M)$ . Let  $\Delta$  be the Laplace-Beltrami operator on  $M$ . If one combines Theorem 1.1 with the usual parametrix construction (cf. [4]) one obtains (arguing as in [14])

**Corollary 1.2.** *Let  $d \geq 4$ ,  $\frac{2(d-1)}{d-3} < q < \infty$ , and let  $I$  be a compact time interval. There is  $C > 0$  such that*

$$\left( \int_I \|e^{it\sqrt{-\Delta}} f\|_{L^q(M)}^q dt \right)^{1/q} \leq C \|f\|_{L_{\alpha}^q(M)}, \quad \alpha = \frac{d-1}{2} - \frac{d}{q},$$

for all  $f \in L_{\alpha}^q(M)$ .

Note that the constant may strongly depend on the choice of  $I$ . There are further regularity improvements in the scale of Triebel-Lizorkin spaces (cf. §3 below); in particular  $L_{\alpha}^q(M)$  can be replaced by the Besov space  $B_{\alpha,q}^q(M)$ .

In §2 we prove a frequency localized version of Theorem 1.1 and combine the estimates corresponding to different frequencies in §3. In §4 we discuss some generalizations in the constant coefficient case.

*Remarks.* In the constant coefficient case one can recover from Theorem 1.1 the space time estimates of [6] which correspond to an endpoint version of Sogge's conjecture in the range given for  $\ell = d - 1$ , see also §4 for other generalizations. For previous partial results on Sogge's conjecture, also in lower dimensions, see the groundbreaking paper of Wolff [20] and the subsequent papers [12], [5].

The case  $\ell = d - 1$  essentially corresponds to the assumption of cinematic curvature in [17]. We use Hörmander's convention for the definition of order, *i.e.*, in view of the different dimensions of  $\mathcal{Z}$  and  $\mathcal{Y}$  operators of class  $I^{\mu-1/4}(\mathcal{Z}, \mathcal{Y}; \mathcal{C})$  correspond to locally finite sums of operators with integral kernels in the standard representation (1) below, involving  $d$  frequency variables and standard symbols of order  $\mu$ . One can use a partition of unity and finite decompositions in the fiber variable to reduce matters to the estimation of an integral operator with compactly supported kernel  $\mathcal{K}$  which is given as an oscillatory integral distribution in the sense of [8]. Namely if  $Z$  is an open set in  $\mathbb{R}^{d+1}$  and  $Y$  is an open set in  $\mathbb{R}^d$  we may assume that

$$(1) \quad \mathcal{K}(z, y) = \int a(z, y, \xi) e^{i(\phi(z, \xi) - \langle y, \xi \rangle)} d\xi$$

where  $a$  is a standard symbol of order  $\mu$ ,  $a$  is supported for  $z, y$  in compact subsets of  $Z$  and  $Y$ , resp., and  $\phi$  is smooth away from the origin and homogeneous of order one with respect to the variable  $\xi$ , and supported in an open set which is conic in  $\xi$ . We then have  $\nabla_z \phi(z, \xi) \neq 0$  for  $\xi \neq 0$  and the mixed second derivative  $(d+1) \times d$  matrix  $\phi''_{z\xi}(z, \xi)$  has rank  $d$ . For fixed  $(z, \xi)$ , if the vector  $u$  is in the cokernel of  $\phi''_{z\xi}(z, \xi)$  then the Hessian matrix  $\nabla_{\xi\xi}^2 \langle u, \nabla_z \phi \rangle(z, \xi)$  has rank at least  $\ell$ , by our curvature assumption.

2. THE FREQUENCY LOCALIZED CASE

By making further localizations, changing variables in  $z$  and  $y$ , and ignoring error terms which are smoothing of high order we may assume that our kernel is given by

$$K(z, y) = \sum_{k=1}^{\infty} 2^{k\mu} K_k(z, y),$$

where

$$(2) \quad K_k(z, y) = \int \chi_k(z, y, 2^{-k}\xi) e^{i(\varphi(z, \xi) - \langle y, \xi \rangle)} d\xi,$$

and the functions  $\chi_k$  are smooth and supported in a compact subset of  $Z \times Y \times \Xi$ . Here  $Z$  is a small neighborhood of the origin in  $\mathbb{R}^{d+1}$ ,  $Y$  is a small neighborhood of the origin in  $\mathbb{R}^d$  and  $\Xi$  is a small neighborhood of the vector  $e_1 := (1, 0, \dots, 0)$  in  $\mathbb{R}^d$ . Moreover

$$(3) \quad \varphi''_{z\xi}(0, e_1) = \begin{pmatrix} I_d \\ 0 \end{pmatrix}$$

$$(4) \quad \text{rank } \nabla_{\xi\xi}^2 \varphi'_{z_{d+1}}(0, e_1) \geq \ell;$$

and in view of the small choice of  $Z, Y, \Xi$  we may assume that for all  $(z, \xi) \in Z \times \Xi$  the gradient  $\varphi'_z(z, \xi)$  is close to  $e_1$ , and, with  $z = (z', z_{d+1})$ , we may assume that  $\varphi''_{z'\xi}(z, \xi)$  is close to the identity matrix  $I_d$  and  $\varphi''_{z_{d+1}\xi}(z, \xi)$  is small. We may further perform a rotation and assume that in coordinates  $\xi = (\xi_1, \xi', \xi'')$  with  $\xi' = (\xi_2, \dots, \xi_{\ell+1})$  we have

$$(5) \quad \text{rank } \nabla_{\xi'\xi'}^2 \varphi'_{z_{d+1}}(0, e_1) = \ell.$$

Finally,  $|\partial_{z,y,\xi}^\alpha \chi_k(z, y, \xi)| \leq C_\alpha$  for any multiindex  $\alpha$ , uniformly in  $k, (z, y, \xi) \in Z \times Y \times \Xi$ .

Let  $T_k f(z) = \int K_k(z, y) f(y) dy$ . Here we prove that the  $L^q(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^{d+1})$  operator norm of  $T_k$  is  $O(2^{k(\frac{d-1}{2} - \frac{d}{q})})$  for  $q > \frac{2\ell}{\ell-2}$ , and in the next section we discuss how to put the estimates for  $T_k$  together. The  $L^\infty$  estimate

$$\|T_k f\|_{L^\infty} \lesssim 2^{k\frac{d-1}{2}} \|f\|_\infty$$

can be found in [16]. By interpolation it is enough to prove

**Theorem 2.1.** *Let  $\ell \geq 3$  and  $q_\ell = \frac{2\ell}{\ell-2}$ . The operator  $T_k$  is of restricted weak type  $(q_\ell, q_\ell)$ , with operator norm*

$$\|T_k\|_{L^{q_\ell, 1}(\mathbb{R}^{d+1}) \rightarrow L^{q_\ell, \infty}(\mathbb{R}^d)} \lesssim 2^{k(d/\ell - 1/2)}.$$

By duality we need to prove the restricted weak type inequality for the adjoint operator  $T_k^*$ , given by

$$T_k^* F(y) = \int \int \chi_k(z, y, 2^{-k}\xi) e^{i(\langle y, \xi \rangle - \varphi(z, \xi))} d\xi F(z) dz;$$

*i.e.* for each measurable set  $E$  contained in  $[-1/2, 1/2]^{d+1}$ ,

$$(6) \quad \|T_k^* \chi_E\|_{L^{p_\ell, \infty}}^{p_\ell} \lesssim 2^{k\frac{2d-\ell}{\ell+2}} |E|, \quad p_\ell = \frac{2\ell}{\ell+2}.$$

The estimate (6) will be derived from the following Proposition 2.2, which is a discretized version of (6) and will be proved in §2.3 for sufficiently small  $\varepsilon > 0$ .

**Proposition 2.2.** *Let  $p_\ell = \frac{2\ell}{\ell+2}$ ,  $\ell \geq 3$ . For  $k > 2$  let  $\mathcal{Z}_k = 2^{-k}\mathbb{Z}^{d+1} \cap [-\varepsilon^2, \varepsilon^2]^{d+1}$  and for each  $\mathfrak{z} \in \mathcal{Z}_k$  we are given a symbol  $a_{k,\mathfrak{z}}$  supported in  $\{\xi : 2^{k-1} < |\xi| < 2^{k+1}, |\frac{\xi}{|\xi|} - e_1| \leq \varepsilon^2\}$  so that*

$$(7) \quad |\partial_\xi^\alpha a_{k,\mathfrak{z}}(\xi)| \leq 2^{-k|\alpha|}, \quad |\alpha| \leq 10d.$$

Define  $S_\mathfrak{z} \equiv S_\mathfrak{z}^k$  by

$$(8) \quad S_\mathfrak{z}(y) = \int a_{k,\mathfrak{z}}(\xi) e^{i\langle y, \xi \rangle - \varphi(\mathfrak{z}, \xi)} d\xi.$$

Then for each  $\mathcal{E} \subset \mathcal{Z}_k$  we have

$$(9) \quad \text{meas}\left(\left\{y \in \mathbb{R}^d : \left|\sum_{\mathfrak{z} \in \mathcal{E}} S_\mathfrak{z}\right| > \alpha\right\}\right) \leq C 2^{k(\frac{d+1}{2}p_\ell-1)} \alpha^{-p_\ell} \#\mathcal{E}.$$

In the following subsections we prove some preparatory  $L^1$  and  $L^2$  estimates, then prove Proposition 2.2, and that Proposition 2.2 implies (6). In §3 we combine the dyadic estimates in Theorem 2.1.

**2.1.  $L^1$  estimates.**  $L^1$ -estimates for the expressions  $S_\mathfrak{z}$  can be found in [16]. In what follows we let  $\Theta_k$  be a maximal  $2^{-k/2}$ -separated set of unit vectors. Using a homogeneous extension of a partition of unity on the sphere one can split

$$a_{k,\mathfrak{z}}(\xi) = \sum_{\theta \in \Theta_k} a_{k,\mathfrak{z},\theta}(\xi)$$

where  $a_{k,\mathfrak{z},\theta}$  is supported on the intersection of the cone  $\{\xi : |\frac{\xi}{|\xi|} - \theta| \leq 2^{-k/2}\}$  with the support of  $a_{k,\mathfrak{z}}$ ; moreover if  $u_i$  are unit vectors perpendicular to  $\theta$  we have the estimates

$$|\langle \theta, \nabla_\xi \rangle^{M_1} \prod_{i=1}^{M_2} \langle u_i, \nabla_\xi \rangle a_{k,\mathfrak{z},\theta}(\xi)| \leq C(M_1, M_2) 2^{-kM_1} 2^{-kM_2/2}$$

whenever  $M_1 + M_2 \leq 10d$ . Let

$$(10) \quad S_{\mathfrak{z},\theta}(y) = \int a_{k,\mathfrak{z},\theta}(\xi) e^{i\langle y, \xi \rangle - \varphi(\mathfrak{z}, \xi)} d\xi.$$

By homogeneity we have

$$(11) \quad \phi''_{\xi\xi}(z, \theta)\theta = 0.$$

Using this observation we get, as in [16], by an integration by parts

$$|S_{\mathfrak{z},\theta}(y)| \leq C_d 2^{k\frac{d+1}{2}} (1 + 2^k |\langle \varphi'_\xi(\mathfrak{z}, \theta) - y, \theta \rangle| + 2^{k/2} |\Pi_{\theta^\perp}(\varphi'_\xi(\mathfrak{z}, \theta) - y)|)^{-10d};$$

here  $\Pi_{\theta^\perp}$  denotes the projection to the orthogonal complement of  $\theta$ . This estimate implies  $\|S_{\mathfrak{z},\theta}\|_1 = O(1)$  and therefore

$$(12) \quad \|S_\mathfrak{z}\|_1 \lesssim 2^{k\frac{d-1}{2}}.$$

Moreover we get for  $1 \leq R \leq 2^k$ ,

$$\int_{\substack{|\Pi_{\theta^\perp}(\varphi'_\xi(\mathfrak{z}, \theta) - y)| \\ \geq (2^{-k}R)^{1/2}}} |S_{\mathfrak{z},\theta}(y)| dy \lesssim \int_{\substack{w' \in \mathbb{R}^{d-1} \\ |w'| \geq (2^{-k}R)^{1/2}}} \frac{2^{k(d-1)/2}}{(1 + 2^{k/2}|w'|)^{10d-2}} dw' \lesssim R^{\frac{1-9d}{2}},$$

and similarly

$$\int_{|\langle \varphi'_\xi(\mathfrak{z}, \theta) - y, \theta \rangle| \geq 2^{-k} R} |S_{\mathfrak{z}, \theta}(y)| dy \lesssim \int_{|w_d| \geq 2^{-k} R} \frac{2^k}{(1 + 2^k |w_d|)^{9d-1}} dw_d \lesssim R^{2-9d}.$$

Now clearly

$$|\varphi'_\xi(z, \theta) - \varphi'_\xi(\tilde{z}, \tilde{\theta})| \lesssim |z - \tilde{z}| + |\theta - \tilde{\theta}|,$$

and by (11) also

$$|\langle \varphi'_\xi(z, \theta) - \varphi'_\xi(\tilde{z}, \tilde{\theta}), \theta \rangle| \lesssim |z - \tilde{z}| + |\theta - \tilde{\theta}|^2.$$

Thus if

$$(13) \quad V_\theta^k(z, R) = \{y : |\langle \varphi'_\xi(z, \theta) - y, \theta \rangle| \leq R2^{-k}, \quad |\Pi_{\theta^\perp}(\varphi'_\xi(z, \theta) - y)| \leq (R2^{-k})^{1/2}\}$$

then the above calculations give

$$(14) \quad \|S_{\tilde{\mathfrak{z}}, \tilde{\theta}}\|_{L^1(\mathbb{R}^d \setminus V_{\theta(\tilde{\mathfrak{z}}, R)}^k)} \leq C(C_1)R^{-4d} \quad \text{if } |\tilde{\mathfrak{z}} - \mathfrak{z}| \leq C_1 R2^{-k}, \quad |\tilde{\theta} - \theta| \leq C_1 (R2^{-k})^{1/2}$$

for  $C_1 \geq 1$ .

**2.2. Estimates for scalar products.** Based on standard calculations for oscillatory integrals ([10], [18], [1], [11], [14]) we prove some estimates for scalar products  $\langle S_{\mathfrak{z}}, S_{\tilde{\mathfrak{z}}} \rangle$ ; these results are closely related to the scalar product estimates in [6]. For the Fourier transforms we have

$$\widehat{S}_{\mathfrak{z}}(\xi) = a_{k, \mathfrak{z}}(\xi) e^{-i\varphi(\mathfrak{z}, \xi)}$$

and

$$(15) \quad (2\pi)^d \langle S_{\mathfrak{z}}, S_{\tilde{\mathfrak{z}}} \rangle = \langle \widehat{S}_{\mathfrak{z}}, \widehat{S}_{\tilde{\mathfrak{z}}} \rangle = \int a_{k, \mathfrak{z}}(\eta) \overline{a_{k, \tilde{\mathfrak{z}}}(\eta)} e^{i(\varphi(\tilde{\mathfrak{z}}, \eta) - \varphi(\mathfrak{z}, \eta))} d\eta$$

$$(16) \quad = 2^{kd} \int b_{k, \mathfrak{z}, \tilde{\mathfrak{z}}}(\xi) e^{i2^k(\varphi(\tilde{\mathfrak{z}}, \xi) - \varphi(\mathfrak{z}, \xi))} d\xi$$

where  $b_{k, \mathfrak{z}, \tilde{\mathfrak{z}}}$  is supported on a subset of diameter  $O(\varepsilon^2)$  of the annulus  $\{|\xi| \approx 1\}$ , near  $e_1$ , with  $\varepsilon$  sufficiently small. We may assume in what follows that  $\mathfrak{z}, \tilde{\mathfrak{z}}$  are in a neighborhood of the origin in  $\mathbb{R}^{d+1}$ , of diameter  $\lesssim \varepsilon^2$ . We split coordinates  $z = (z', z_{d+1})$ , take advantage of (3) and get

$$|\varphi'_\xi(\mathfrak{z}, \xi) - \varphi'_\xi(\tilde{\mathfrak{z}}, \xi)| \geq c|\mathfrak{z}' - \tilde{\mathfrak{z}}'| - C\varepsilon|\mathfrak{z}_{d+1} - \tilde{\mathfrak{z}}_{d+1}|$$

and after an integration by parts we get

$$(17) \quad |\langle S_{\mathfrak{z}}, S_{\tilde{\mathfrak{z}}} \rangle| \lesssim \frac{2^{kd}}{(1 + 2^k |\mathfrak{z} - \tilde{\mathfrak{z}}|)^{9d}} \quad \text{if } |\mathfrak{z}' - \tilde{\mathfrak{z}}'| \geq C_1 \varepsilon |\mathfrak{z}_{d+1} - \tilde{\mathfrak{z}}_{d+1}|.$$

For  $s \in [0, 1]$  set  $\mathfrak{z}_s = \mathfrak{z} + s(\tilde{\mathfrak{z}} - \mathfrak{z})$ . If

$$|\mathfrak{z}' - \tilde{\mathfrak{z}}'| \leq C_2 \varepsilon |\mathfrak{z}_{d+1} - \tilde{\mathfrak{z}}_{d+1}|$$

(with suitable  $C_1 \ll C_2 \ll \varepsilon^{-1}$ ) we consider

$$\frac{\varphi(\tilde{\mathfrak{z}}, \xi) - \varphi(\mathfrak{z}, \xi)}{\tilde{\mathfrak{z}}_{d+1} - \mathfrak{z}_{d+1}} = \int_0^1 \left[ \varphi'_{z_{d+1}}(\mathfrak{z}_s, \xi) + \left\langle \frac{\tilde{\mathfrak{z}}' - \mathfrak{z}'}{\tilde{\mathfrak{z}}_{d+1} - \mathfrak{z}_{d+1}}, \varphi'_{z'}(\mathfrak{z}_s, \xi) \right\rangle \right] ds.$$

Note that  $\frac{\varphi(\tilde{\mathfrak{z}}, \xi) - \varphi(\mathfrak{z}, \xi)}{\tilde{\mathfrak{z}}_{d+1} - \mathfrak{z}_{d+1}}$  is a small perturbation of  $\varphi'_{z_{d+1}}(0, \xi)$  if  $\varepsilon$  is sufficiently small. We apply the method of stationary phase (with parameters, [8]) in the  $\xi'$ -variables, using (5). This yields

$$|\langle S_{\mathfrak{z}}, S_{\tilde{\mathfrak{z}}} \rangle| \lesssim \frac{2^{kd}}{(1 + 2^k |\tilde{\mathfrak{z}}_{d+1} - \mathfrak{z}_{d+1}|)^{\ell/2}}, \quad \text{if } |\mathfrak{z}' - \tilde{\mathfrak{z}}'| \leq C_2 \varepsilon |\mathfrak{z}_{d+1} - \tilde{\mathfrak{z}}_{d+1}|,$$

and combining this with (17) we get

$$(18) \quad |\langle S_{\mathfrak{z}}, S_{\tilde{\mathfrak{z}}} \rangle| \lesssim \frac{2^{kd}}{(1 + 2^k |\tilde{\mathfrak{z}} - \mathfrak{z}|)^{\ell/2}}$$

whenever  $|\mathfrak{z} - \tilde{\mathfrak{z}}| = O(\varepsilon^2)$ .

**2.3. Proof of Proposition 2.2.** If  $\alpha \leq 2^{k \frac{d+1}{2}}$  then the desired inequality follows from (12). Indeed by Tshebyshev's inequality the left hand side of (9) is  $\lesssim \alpha^{-1} 2^{k(d-1)/2} \#\mathcal{E}$  which is dominated by the right hand side of (9) if  $\alpha \leq 2^{k \frac{d+1}{2}}$ .

In what follows we shall therefore assume that  $\alpha > 2^{k \frac{d+1}{2}}$  and set

$$(19) \quad u_k(\alpha) := (\alpha 2^{-k \frac{d+1}{2}})^{p_\ell} > 1.$$

The argument is a variant of one in [6]; it is based on a Calderón-Zygmund type decomposition at height  $u_k(\alpha)$  where volume is replaced by diameter.

By the usual Vitali procedure there is a finite (possibly empty) family  $\mathfrak{B}^k$  of disjoint balls so that

$$(20) \quad u_k(\alpha) 2^k \text{diam}(B) \leq \#(\mathcal{E} \cap B) \quad \text{for } B \in \mathfrak{B}^k;$$

moreover if we remove the balls in  $\mathfrak{B}^k$  and set

$$\mathcal{E}_* = \mathcal{E} \setminus \bigcup_{B \in \mathfrak{B}^k} B,$$

then

$$(21) \quad \#(\mathcal{E}_* \cap B) \leq C_d u_k(\alpha) 2^k \text{diam}(B) \quad \text{for every ball } B.$$

Since  $\mathcal{E} \subset \mathcal{Z}_k$  which is  $2^{-k}$ -separated, we may assume that  $\text{diam}(B) \geq 2^{-k}$  if  $B \in \mathfrak{B}^k$ .

We need to establish the following two inequalities:

$$(22) \quad \text{meas} \left( \left\{ y \in \mathbb{R}^d : \left| \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} S_{\mathfrak{z}} \right| > \alpha/2 \right\} \right) \leq C 2^{k(\frac{d+1}{2} p_\ell - 1)} \alpha^{-p_\ell} \#\mathcal{E},$$

$$(23) \quad \text{meas} \left( \left\{ y \in \mathbb{R}^d : \left| \sum_{\mathfrak{z} \in \mathcal{E}_*} S_{\mathfrak{z}} \right| > \alpha/2 \right\} \right) \leq C 2^{k(\frac{d+1}{2} p_\ell - 1)} \alpha^{-p_\ell} \#\mathcal{E}.$$

*Proof of (22).* We first form an exceptional set as follows. Let  $z_B$  denote the center of a ball  $B \in \mathfrak{B}^k$  and let  $R_B = 10d2^k \text{diam}(B) \gtrsim 1$ . Let  $\Theta(k, B)$  be a maximal  $C_1(2^{-k} R_B)^{1/2}$  separated subset of  $S^{d-1}$ . Here  $C_1$  is the constant in (14). Define (using the notation in (13))

$$(24) \quad \mathcal{V}^k = \bigcup_{B \in \mathfrak{B}^k} \bigcup_{\vartheta \in \Theta(k, B)} V_{\vartheta}^k(z_B, R_B).$$

Observe that  $\text{meas}(V_{\vartheta}^k(z_B, R_B))$  is  $O((R_B 2^{-k})^{(d+1)/2})$  and  $\#\Theta(k, B) = O((2^k R_B^{-1})^{(d-1)/2})$ . Thus

$$\begin{aligned} \text{meas}(\mathcal{V}^k) &\lesssim \sum_{B \in \mathfrak{B}^k} \sum_{\vartheta \in \Theta(k, B)} \text{meas}(V_{\vartheta}^k(z_B, R_B)) \lesssim \sum_{B \in \mathfrak{B}^k} R_B 2^{-k} \\ &\lesssim \sum_{B \in \mathfrak{B}^k} \text{diam}(B) \lesssim \sum_{B \in \mathfrak{B}^k} 2^{-k} \frac{\#(\mathcal{E} \cap B)}{u_k(\alpha)} \lesssim 2^{k(-1 + \frac{d+1}{2} p_\ell)} \alpha^{-p_\ell} \#\mathcal{E}, \end{aligned}$$

by the disjointness of the balls in  $\mathfrak{B}^k$ , (20), and the definition of  $u_k(\alpha)$ .

To conclude the proof of (22) we have to estimate the contribution in the complement of  $\mathcal{V}^k$ . For this we bound

$$\text{meas}\left(\left\{y \in \mathbb{R}^d \setminus \mathcal{V}^k : \left| \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} S_{\mathfrak{z}} \right| > \frac{\alpha}{2} \right\}\right) \lesssim \alpha^{-1} \left\| \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} S_{\mathfrak{z}} \right\|_{L^1(\mathbb{R}^d \setminus \mathcal{V}^k)}.$$

Now fix  $B$ . For every  $\theta \in \Theta_k$  we may choose a  $\vartheta = \vartheta_B(\theta) \in \Theta(k, B)$  so that  $|\vartheta_B(\theta) - \theta| \leq C_1(R_B 2^{-k})^{-1/2}$ . Recalling  $S_{\mathfrak{z}} = \sum_{\theta \in \Theta_k} S_{\mathfrak{z}, \theta}$ , we see

$$\begin{aligned} & \left\| \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} S_{\mathfrak{z}} \right\|_{L^1(\mathbb{R}^d \setminus \mathcal{V}^k)} \\ & \leq \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} \sum_{\theta \in \Theta_k} \|S_{\mathfrak{z}, \theta}\|_{L^1(\mathbb{R}^d \setminus V_{\vartheta_B(\theta)}^k(z_B, R_B))} \\ & \lesssim \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} \sum_{\theta \in \Theta_k} R_B^{-4d} \lesssim \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} 2^{k(d-1)/2} R_B^{-4d}. \end{aligned}$$

For the second inequality we use (14) and the last one follows from  $\#\Theta_k = O(2^{k(d-1)/2})$ . Now we note that  $2^{k \frac{d-1}{2}} \alpha^{-1} = 2^{k(-1 + \frac{d+1}{2} p_\ell)} \alpha^{-p_\ell} u_k(\alpha)^{1 - \frac{1}{p_\ell}}$ . Thus

$$\begin{aligned} & \text{meas}\left(\left\{y \in \mathbb{R}^d \setminus \mathcal{V}^k : \left| \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} S_{\mathfrak{z}} \right| > \frac{\alpha}{2} \right\}\right) \lesssim \alpha^{-1} \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} 2^{k(d-1)/2} R_B^{-4d} \\ & \lesssim 2^{k(-1 + \frac{d+1}{2} p_\ell)} \alpha^{-p_\ell} u_k(\alpha)^{1 - \frac{1}{p_\ell}} \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} R_B^{-4d}. \end{aligned}$$

By (20) we have for  $B \in \mathfrak{B}^k$

$$u_k(\alpha) \lesssim \frac{\#(\mathcal{E} \cap B)}{R_B} \lesssim R_B^d$$

and therefore

$$\begin{aligned} & \text{meas}\left(\left\{y \in \mathbb{R}^d \setminus \mathcal{V}^k : \left| \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} S_{\mathfrak{z}} \right| > \frac{\alpha}{2} \right\}\right) \\ & \lesssim 2^{k(-1 + \frac{d+1}{2} p_\ell)} \alpha^{-p_\ell} u_k(\alpha)^{1 - \frac{1}{p_\ell} - 4} \sum_{B \in \mathfrak{B}^k} \#(\mathcal{E} \cap B) \lesssim 2^{k(-1 + \frac{d+1}{2} p_\ell)} \alpha^{-p_\ell} \#\mathcal{E} \end{aligned}$$

since  $u_k(\alpha) \geq 1$  and  $R_B \gtrsim 1$ . □

*Proof of (23).* We check from (19) that

$$2^{kd} u_k(\alpha)^{2/\ell} \alpha^{-2} = 2^{k(-1 + \frac{d+1}{2} p_\ell)} \alpha^{-p_\ell}.$$

Thus by Tshebyshev's inequality it suffices to prove

$$(25) \quad \left\| \sum_{\mathfrak{z} \in \mathcal{E}_*} S_{\mathfrak{z}} \right\|_2^2 \lesssim 2^{kd} u_k(\alpha)^{2/\ell} \#\mathcal{E}_*.$$

We set

$$\begin{aligned} L &:= u_k(\alpha)^{2/\ell}, \\ I(n, L) &:= [n2^{-k}L, (n+1)2^{-k}L), \\ \mathcal{E}(n, L) &:= \{\mathfrak{z} \in \mathcal{E}_* : \mathfrak{z}_{d+1} \in I(n, L)\}, \\ \mathfrak{S}_n &:= \sum_{\mathfrak{z} \in \mathcal{E}(n, L)} S_{\mathfrak{z}}. \end{aligned}$$

Now

$$\left\| \sum_{\mathfrak{z} \in \mathcal{E}_*} S_{\mathfrak{z}} \right\|_2^2 \lesssim \sum_n \sum_{\tilde{n}: |n-\tilde{n}| \leq 4} \langle \mathfrak{S}_n, \mathfrak{S}_{\tilde{n}} \rangle + \sum_n \sum_{\tilde{n}: |n-\tilde{n}| > 4} \langle \mathfrak{S}_n, \mathfrak{S}_{\tilde{n}} \rangle =: I + II.$$

For  $I$  we use the Schwarz inequality and then (17) to get

$$\begin{aligned} |I| &\lesssim \sum_n \|\mathfrak{S}_n\|_2^2 = \sum_n \left\| \sum_{\mathfrak{z}_{d+1} \in I(n, L)} \sum_{\mathfrak{z}'} S_{(\mathfrak{z}', \mathfrak{z}_{d+1})} \right\|_2^2 \\ &\lesssim L \sum_n \sum_{\mathfrak{z}_{d+1} \in I(n, L)} \left\| \sum_{\substack{\mathfrak{z}': \\ (\mathfrak{z}', \mathfrak{z}_{d+1}) \in \mathcal{E}_*}} S_{(\mathfrak{z}', \mathfrak{z}_{d+1})} \right\|_2^2 \\ &\lesssim L \sum_n \sum_{\mathfrak{z}_{d+1} \in I(n, L)} \sum_{\substack{\mathfrak{z}': \\ (\mathfrak{z}', \mathfrak{z}_{d+1}) \in \mathcal{E}}} \sum_{\tilde{\mathfrak{z}}'} 2^{kd} (1 + 2^k |\mathfrak{z}' - \tilde{\mathfrak{z}}'|)^{-8d} \lesssim L 2^{kd} \#\mathcal{E}. \end{aligned}$$

For  $II$  we use (18) and estimate

$$|II| \lesssim \sum_{\mathfrak{z} \in \mathcal{E}_*} \sum_{\substack{\tilde{\mathfrak{z}} \in \mathcal{E}_* \\ |\mathfrak{z}_{d+1} - \tilde{\mathfrak{z}}_{d+1}| \geq 2^{-k}L}} |\langle S_{\mathfrak{z}}, S_{\tilde{\mathfrak{z}}} \rangle| \lesssim \sum_{\mathfrak{z} \in \mathcal{E}_*} \sum_{\substack{\tilde{\mathfrak{z}} \in \mathcal{E}_* \\ |\mathfrak{z}_{d+1} - \tilde{\mathfrak{z}}_{d+1}| \geq 2^{-k}L}} \frac{2^{kd}}{(1 + 2^k |\mathfrak{z} - \tilde{\mathfrak{z}}|)^{\ell/2}}.$$

By (21) we have for  $R \geq 1$  and fixed  $\mathfrak{z}$

$$\sum_{\substack{\tilde{\mathfrak{z}} \in \mathcal{E}_* \\ 2^{-k}R \leq |\mathfrak{z} - \tilde{\mathfrak{z}}| \leq 2^{1-k}R}} (1 + 2^k |\mathfrak{z} - \tilde{\mathfrak{z}}|)^{-\ell/2} \lesssim u_k(\alpha) R^{1-\frac{\ell}{2}}$$

and since  $\ell/2 > 1$  we get (after setting  $R = 2^m$  and summing over  $m$  with  $2^m \geq L$ )

$$|II| \lesssim 2^{kd} u_k(\alpha) L^{1-\frac{\ell}{2}} \#\mathcal{E}.$$

Hence

$$I + II \lesssim 2^{kd} \#\mathcal{E} (L + u_k(\alpha) L^{1-\frac{\ell}{2}})$$

and with the optimal choice of  $L = [u_k(\alpha)]^{2/\ell}$  we obtain (25).  $\square$

**2.4. Proof that Proposition 2.2 implies (6).** We shall first assume that in (2)

$$(26) \quad \chi_k(z, y, 2^{-k}\xi) = \eta_k(z, 2^{-k}\xi) \chi_{\circ}(y)$$

where  $\chi_{\circ} \in C_c^{\infty}(\mathbb{R}^d)$  is supported on a small neighborhood of the origin but so that  $\chi_{\circ}(y) = 1$  for  $y \in E$ ; moreover  $\eta_k$  is compactly supported in a set of diameter  $O(\varepsilon^2)$  near  $(z, \xi) = (0, e_1)$ , and the derivatives of  $\eta_k$  up to order  $10d$  are uniformly bounded.

Let  $Q_{\mathfrak{z}} = \prod_{i=1}^{d+1} [\mathfrak{z}_i, \mathfrak{z}_i + 2^{-k}]$ . For  $m \geq 0$  let

$$(27) \quad \begin{aligned} \mathcal{E}_m &= \{\mathfrak{z} \in \mathcal{Z}_k : 2^{-k(d+1)-m-1} < |Q_{\mathfrak{z}} \cap E| \leq 2^{-k(d+1)-m}\}, \\ E_m &= \bigcup_{\mathfrak{z} \in \mathcal{E}_m} Q_{\mathfrak{z}} \cap E. \end{aligned}$$

And we also set

$$\begin{aligned} a_{k,\mathfrak{z},m}(\xi) &= 2^{m+(k+1)d} \int_{Q_{\mathfrak{z}} \cap E_m} \eta_k(z, 2^{-k}\xi) e^{i(\varphi(\mathfrak{z},\xi) - \varphi(z,\xi))} dz, \\ S_{\mathfrak{z},m}(y) &= \int a_{k,\mathfrak{z},m}(\xi) e^{i\langle y,\xi \rangle - \varphi(\mathfrak{z},\xi)} d\xi. \end{aligned}$$

Then it follows that

$$T_k^* \chi_E(y) = \sum_{m=0}^{\infty} 2^{-m-(k+1)d} S_{\mathfrak{z},m}(y).$$

Since  $\partial_{\xi}^{\alpha}(\varphi(\mathfrak{z},\xi) - \varphi(z,\xi)) = O(2^{-k})$  for any multiindex  $\alpha$  it is easy to see that  $a_{\cdot,\cdot,m}$  satisfies (7) uniformly in  $m$ . Hence, the result of Proposition 2.2 can be applied to  $\sum_{\mathfrak{z} \in \mathcal{E}_m} S_{\mathfrak{z},m}(y)$  and we get uniform bounds. Thus

$$\left\| \sum_{\mathfrak{z} \in \mathcal{E}_m} S_{\mathfrak{z},m} \right\|_{L^{p_{\ell},\infty}} \lesssim 2^{k(\frac{d+1}{2} - \frac{1}{p_{\ell}})} (\#\mathcal{E}_m)^{1/p_{\ell}} \lesssim 2^{k(\frac{d+1}{2} - \frac{1}{p_{\ell}})} 2^{m/p_{\ell}} (2^{k(d+1)} \text{meas}(E_m))^{1/p_{\ell}}$$

with implicit constants independent of  $m$ . For the second inequality we use

$$\#\mathcal{E}_m \lesssim 2^m 2^{k(d+1)} \text{meas}(E_m)$$

which follows from (27). Consequently we get

$$\begin{aligned} \|T_k^* \chi_E\|_{L^{p_{\ell},\infty}} &\lesssim \sum_{m=0}^{\infty} 2^{-m-(k+1)d} \|S_{\mathfrak{z},m}\|_{L^{p_{\ell},\infty}} \\ &\lesssim \sum_{m=0}^{\infty} 2^{-m(1-\frac{1}{p_{\ell}})} 2^{k(\frac{d}{p_{\ell}} - \frac{d+1}{2})} (\text{meas}(E_m))^{1/p_{\ell}} \lesssim 2^{k(\frac{d}{p_{\ell}} - \frac{d+1}{2})} (\text{meas}(E))^{1/p_{\ell}} \end{aligned}$$

which is the desired estimate.

Finally we have to remove the assumption (26). Here one uses Fourier series in  $y$  and expands  $\chi_k(z, y, 2^{-k}\xi) = \sum_{\nu \in \mathbb{Z}^d} c_{k,\nu} \eta_{k,\nu}(z, 2^{-k}\xi) e^{i\langle y,\nu \rangle}$  where the functions  $\eta_{k,\nu}(z, 2^{-k}\xi)$  are as before but now with a bound that decays fast in  $\nu$ . We note that multiplication with  $e^{i\langle y,\nu \rangle}$  does not affect the  $L^{p_{\ell},1}$  norm, apply the previous bounds to the summands and sum in  $\nu$  using the rapid decay in  $\nu$ .  $\square$

### 3. COMBINING THE FREQUENCY LOCALIZED PIECES

We now combine the previous estimates on the operators  $T_k$  and prove the following result in Triebel-Lizorkin spaces  $F_{a,s}^q$ . Recall ([19]) that if  $\{L_k\}_{k=0}^{\infty}$  is a standard inhomogeneous dyadic frequency decomposition then the norm  $\|f\|_{F_{a,s}^q}$  can be defined as the  $L^q(\ell^s)$  norm of the sequence  $\{2^{ka} L_k f\}$ . In view of the embeddings  $L^q = F_{0,2}^q \subset F_{0,q}^q$  for  $q \geq 2$ ,  $L^q \supset F_{0,r}^q$  for  $r \leq 2$  the following result sharpens Theorem 1.1. For the case  $r \geq 1$  one could argue by duality and follow [6] but we shall rely on a result in [15] which gives an estimate for all  $r > 0$ .

**Theorem 3.1.** *Let  $\ell \geq 3$ , let  $\mathcal{Z}, \mathcal{Y}$  be coordinate patches in  $\mathbb{R}^{d+1}, \mathbb{R}^d$ , resp., let  $\mathcal{F} \in I^{\mu-1/4}(\mathcal{Z}, \mathcal{Y}; \mathbb{C})$ , with Schwartz kernel compactly supported in  $\mathcal{Z} \times \mathcal{Y}$ , and let  $\mathcal{C}$  satisfy hypothesis  $\mathcal{H}(\ell)$ . Suppose  $\frac{2\ell}{\ell-2} < q < \infty$ ,  $a = \mu + b + d(1/2 - 1/q) - 1/2$  and  $r > 0$ . Then  $\mathcal{F}$  maps  $F_{a,q}^q(\mathbb{R}^d)$  boundedly to  $F_{b,r}^q(\mathbb{R}^{d+1})$ .*

We state (a slight variant of) the result from [15]. In this setting one is given operators  $\mathcal{T}_k$  defined on the Schwartz space  $\mathcal{S}(\mathbb{R}^{d_1})$ ,

$$\mathcal{T}_k f(z) = \int \mathcal{K}_k(z, y) f(y) dy, \quad z \in \mathbb{R}^{d_2}$$

and each  $\mathcal{K}_k$  is continuous and bounded. Let  $\zeta \in \mathcal{S}(\mathbb{R}^{d_2})$  and  $\zeta_k(z) = 2^{kd_2} \zeta(2^k z)$ , and define  $P_k g = \zeta_k * g$ .

**Theorem 3.2** ([15]). *Let  $d_1 \leq d_2$ ,  $0 < \gamma < d_2$ ,  $\varepsilon > 0$ ,  $1 < q_0 < q < \infty$ , and assume*

$$(28) \quad \sup_{k>0} 2^{k\gamma/q_0} \|\mathcal{T}_k\|_{L^p(\mathbb{R}^{d_1}) \rightarrow L^p(\mathbb{R}^{d_2})} < \infty,$$

$$(29) \quad \sup_{k>0} \|\mathcal{T}_k\|_{L^\infty(\mathbb{R}^{d_1}) \rightarrow L^\infty(\mathbb{R}^{d_2})} < \infty.$$

Furthermore assume that for each cube  $Q$  there is a measurable set  $\mathcal{W}_Q \subset \mathbb{R}^{d_2}$  so that

$$(30) \quad \text{meas}(\mathcal{W}_Q) \leq C \max\{|Q|^{1-\gamma/d_2}, |Q|\}$$

and there is  $\delta > 0$  such that for every  $k \in \mathbb{N}$  and every cube  $Q$  with  $2^k \text{diam}(Q) \geq 1$

$$(31) \quad \sup_{x \in Q} \int_{\mathbb{R}^{d_2} \setminus \mathcal{W}_Q} |\mathcal{K}_k(x, y)| dy \leq C \max\{(2^k \text{diam}(Q))^{-\delta}, 2^{-k\delta}\}.$$

Then for  $q_0 < q < \infty$ ,  $r > 0$

$$\left\| \left( \sum_k 2^{k\gamma r/q} |P_k \mathcal{T}_k f_k|^r \right)^{1/r} \right\|_q \lesssim \left\| \left( \sum_k \|f_k\|_q^q \right)^{1/q} \right\|_q.$$

This (or a slightly sharper version) was formulated in [15] only for the case  $d_1 = d_2$ , but the result there implies the version cited above. Indeed if  $d_1 < d_2$  we can define an operator  $\tilde{\mathcal{T}}_k$  on functions  $F$  on  $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2-d_1}$  by

$$\tilde{\mathcal{T}}_k F(z) = \iint K_k(z, y) \chi(w) F(y, w) dy dw$$

where  $\chi$  is a nontrivial  $C_c^\infty(\mathbb{R}^{d_2-d_1})$  function. The assumptions on  $\mathcal{T}_k$  imply the corresponding assumptions on  $\tilde{\mathcal{T}}_k$ , by Minkowski's and Hölder's inequalities. Thus the equidimensional case in [15] may be applied and if in the conclusion we specialize to tensor products,  $F(y, w) = f(y)\chi_1(w)$  we get the above generalization.

In order to prepare for our application of Theorem 3.2 we let  $T_k f(z) = \int K_k(z, y) f(y) dy$  with  $K_k$  as in (2). Let  $\beta_0$  be a  $C^\infty$ -function supported in  $\{\eta \in \mathbb{R}^{d+1} : |\eta| \leq 3/2\}$  so that  $\beta_0(\eta) = 1$  for  $|\eta| \leq 1$ ; and, for  $k \geq 1$  let  $\beta_k(\eta) = \beta_0(2^{-k}(\eta)) - \beta_0(2^{1-k}(\eta))$ . Define  $L_k$  on functions on  $\mathbb{R}^{d+1}$  by  $\widehat{L_k g} = \beta_k(\eta) \widehat{g}(\eta)$ . We use calculations in [8] and first observe that there is a constant  $A_0 > 1$  so that

$$(32) \quad \|L_{\tilde{k}} T_k\|_{L^q \rightarrow L^q} \leq C_N \min\{2^{-kN}, 2^{-\tilde{k}N}\} \quad \text{if } |k - \tilde{k}| \geq A_0.$$

This follows from the assumption that  $\mathcal{C}$  does not meet the zero sections, which, by homogeneity implies that  $c_1|\xi| \leq |\varphi'_z(z, \xi)| \leq C_1|\xi|$ . The kernel  $\mathcal{K}_{k\tilde{k}}$  of  $L_{\tilde{k}}T_k$  is given by

$$\mathcal{K}_{k\tilde{k}}(z, y) = \iiint \beta_{\tilde{k}}(\eta)\chi_k(z, y, 2^{-k}\xi)e^{i((z-w, \eta)+\varphi(w, \xi)-(y, \xi))}dw d\eta d\xi$$

and if  $k - \tilde{k}$  are sufficiently large then  $|\eta + \varphi'_w(w, \xi)| \approx \max\{|\xi|, |\eta|\}$  on the support of the amplitude. Thus in this case we may use an integration by parts in  $w$  (followed by an integration by parts in  $\xi$  when  $z$  is large) to show that the kernels  $\mathcal{K}_{k\tilde{k}}$  of  $L_{\tilde{k}}T_k$  satisfy the estimate

$$|\mathcal{K}_{k\tilde{k}}(z, y)| \leq C_N 2^{-kN} (1 + |z|)^{-N}$$

and vanish for  $y$  in the complement of a fixed compact set. Thus (32) follows. Similarly if  $\{\mathcal{L}_k\}_{k=0}^\infty$  is the corresponding frequency decomposition in  $\mathbb{R}^d$  we also see that  $T_k\mathcal{L}_{\tilde{k}}$  has  $L^q(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^{d+1})$  operator norm  $\leq C_N \min\{2^{-kN}, 2^{-\tilde{k}N}\}$  if  $|k - \tilde{k}| > 4$ .

From these preliminary remarks it follows quickly that for the proof of Theorem 3.1 it suffices to prove the inequalities

$$\left\| \left( \sum_k 2^{kbr} |2^{k\mu} L_{k+i_1} T_k \mathcal{L}_{k+i_2} g|^r \right)^{1/r} \right\|_q \lesssim \left\| \left( \sum_k 2^{kaq} \|\mathcal{L}_{k+i_2} g\|_q^q \right)^{1/q} \right\|_q, \quad |i_1| \leq A_0, \quad |i_2| \leq 4$$

with  $a = \mu + b + d(1/2 - 1/q) - 1/2$ . Setting  $\mathcal{T}_k = 2^{-k\frac{d-1}{2}} T_k$  and  $f_k = 2^{ka} \mathcal{L}_{k+i_2} g$ , the preceding inequality follows from

$$(33) \quad \left\| \left( \sum_k 2^{k\frac{d}{q}r} |L_{k+i_1} \mathcal{T}_k f_k|^r \right)^{1/r} \right\|_q \lesssim \left\| \left( \sum_k \|f_k\|_q^q \right)^{1/q} \right\|_q, \quad |i_1| \leq A_0,$$

for  $q_0 < q < \infty$ , where  $q_0 > \frac{2\ell}{\ell-2}$ . This in turn follows from an application of Theorem 3.2 with  $d_1 = d$ ,  $d_2 = d + 1$ ,  $\gamma = d$ , and  $P_k = L_{k+i_1}$ . The hypothesis (29) follows from the calculations in [16] (*cf.* also §2.1 above). The hypothesis (28) for  $\frac{2\ell}{\ell-2} < q < \infty$  follows from Theorem 2.1 and (29) by interpolation.

If  $\text{diam}(Q) > \varepsilon$  then  $\mathcal{W}_Q$  is simply an expanded cube and (31) follows by the support assumption of  $K_k$ . If  $\text{diam}(Q) < \varepsilon$  the exceptional sets are formed as in §2.1. For  $\theta \in S^{d-1}$  we set

$$W_\theta(z_Q, C) = \{y : |\langle \varphi'_\xi(z_Q, \theta) - y, \theta \rangle| \leq C \text{diam}(Q), \quad |\Pi_{\theta^\perp}(\varphi'_\xi(z, \theta) - y)| \leq C(\text{diam}(Q))^{1/2}\}$$

and if  $\Theta_Q$  is a maximal set of  $(\text{diam}(Q))^{1/2}$  separated unit vectors we set

$$\mathcal{W}_Q = \bigcup_{\theta \in \Theta_Q} W_\theta(z_Q, C).$$

Then the measure of  $\mathcal{W}_Q$  is  $O(\text{diam}(Q)) = O(|Q|^{1-\frac{d}{d+1}})$  so that (30) holds. The hypothesis (31) (even with large  $\delta$ ) holds by the calculations in §2.1 (*cf.* (14)).

#### 4. REMARKS ON THE CONSTANT COEFFICIENT CASE

We now let  $\rho$  be a  $C^\infty(\mathbb{R}^d \setminus \{0\})$  function which is homogeneous of degree 1, so that  $\rho(\xi) \neq 0$  for  $\xi \neq 0$ . We are interested in space time estimates for  $U_t \equiv e^{-it\rho(D)}$  defined by

$$\widehat{U_t f}(\xi) = e^{it\rho(\xi)} \widehat{f}(\xi)$$

and obtain a result under a decay assumption for the Fourier transform of surface carried measure on

$$\Sigma_\rho = \{\xi : \rho(\xi) = 1\}.$$

**Theorem 4.1.** *Let  $\kappa > 1$  and let  $\rho$  be as above such that the surface measure  $d\sigma$  of  $\Sigma_\rho$  satisfies*

$$(34) \quad \sup_{\xi} (1 + |\xi|)^\kappa |\widehat{d\sigma}(\xi)| < \infty.$$

Let  $\frac{2\kappa}{\kappa-1} < q < \infty$  and let  $I$  be a compact time interval. There is  $C > 0$  such that

$$\left( \int_I \|e^{it\rho(D)} f\|_{L^q(\mathbb{R}^d)}^q dt \right)^{1/q} \leq C \|f\|_{B_{\alpha,q}^q(\mathbb{R}^d)}, \quad \alpha = \frac{d-1}{2} - \frac{d}{q},$$

for all  $f \in B_{\alpha,q}^q(\mathbb{R}^d)$ .

*Remark.* Under the assumption that  $\Sigma_\rho$  has nonvanishing curvature everywhere this follows from Theorem 3.1 above. We note that in this particular case a weaker result with  $\alpha > (d-1)/2 - d/q$  is already in [7].

*Sketch of proof.* We will assume that  $I = [-1, 1]$ , as one can use rescaling to reduce to this case. Fix  $k$  and define  $S_{x,t} \equiv S_{x,t}^k$  by

$$S_{x,t}(y) = \int e^{i\langle x-y, \xi \rangle + it\rho(\xi)} \chi(2^{-k}\xi) d\xi$$

and as before we may assume that the support of  $\chi$  has diameter  $\leq \varepsilon^2$ , for sufficiently small  $\varepsilon > 0$ , and is contained in  $\{1/2 < |\xi| \leq 2\}$ . Fix  $\xi_\circ \in \Sigma_\rho$  and  $\rho_\circ > 0$  so that  $\rho_\circ \xi_\circ \in \text{supp}(\chi)$ . Let  $u \mapsto \Xi(u)$  be a parametrization of  $\Sigma_\rho$  near  $\xi_\circ$  with parameter  $u \in \mathbb{R}^{d-1}$  near  $u_\circ$ , and  $\Xi(u_\circ) = \xi_\circ$ . Let  $\mathbf{n}_\circ$  be the outer normal unit vector to  $\Sigma_\rho$  at  $\xi_\circ$ . Let  $\Gamma$  be the cone formed by the  $(\rho\Xi(u), \rho)$  with  $\rho > 0$  and  $u$  near  $u_\circ$ . Let  $\vec{N}_\circ = \mathbf{n}_\circ - \langle \xi_\circ, \mathbf{n}_\circ \rangle \vec{e}_{d+1}$ , which is a normal vector to  $\Gamma$  at  $(\rho\Xi(u_\circ), \rho)$ . By finite decompositions of  $\chi$  we may further assume that

$$S_{x,t}(y) = \int \beta(2^{-k}\xi) e^{i\langle (x-y, \xi) + it\rho(\xi) \rangle} d\xi$$

where  $\beta \in C_c^\infty$  supported in an  $\varepsilon^2$ -neighborhood of  $\rho_\circ \Xi(u_\circ)$ .

The proof of Theorem 4.1 is a straightforward variant of the proof of Theorem 3.1 once we have established the two appropriate replacements for the scalar product bounds (18) and (17), namely

$$(35) \quad |\langle S_{x,t}, S_{\tilde{x}, \tilde{t}} \rangle| \lesssim 2^{kd} (1 + 2^k |x - \tilde{x}| + 2^k |t - \tilde{t}|)^{-\kappa},$$

and a better estimate when  $(x - \tilde{x}, t - \tilde{t})$  is orthogonal (or near orthogonal) to  $\vec{N}_\circ$ :

$$(36) \quad |\langle S_{x,t}, S_{\tilde{x}, \tilde{t}} \rangle| \leq C_M 2^{kd} (1 + 2^k |x - \tilde{x}| + 2^k |t - \tilde{t}|)^{-M}$$

if  $|\langle x - \tilde{x}, \mathbf{n}_\circ \rangle - (t - \tilde{t}) \langle \xi_\circ, \mathbf{n}_\circ \rangle| \leq \epsilon_\circ |x - \tilde{x}, t - \tilde{t}|$

for a small  $\epsilon_\circ > 0$ .

As before  $(2\pi)^d \langle S_{x,t}, S_{\tilde{x}, \tilde{t}} \rangle = \langle \widehat{S}_{x,t}, \widehat{S}_{\tilde{x}, \tilde{t}} \rangle$ . We scale and then use generalized polar coordinates  $\xi = \rho\Xi(u)$  to write

$$(37) \quad \langle \widehat{S}_{x,t}, \widehat{S}_{\tilde{x}, \tilde{t}} \rangle = 2^{kd} \iint b(\rho, u) e^{i2^k \rho (\langle x - \tilde{x}, \Xi(u) \rangle + t - \tilde{t})} du d\rho$$

where  $b$  is smooth and supported near  $(\rho_\circ, u_\circ)$ . Now for any  $\chi \in C_c^\infty$  we have  $|\chi \widehat{d\sigma}(\xi)| \lesssim (1 + |\xi|)^\kappa$ , by assumption (34). We apply this to the inner integral of (37) and obtain

$$(38) \quad |\langle S_{x,t}, S_{\tilde{x}, \tilde{t}} \rangle| \lesssim 2^{kd} (1 + 2^k |x - \tilde{x}|)^{-\kappa}.$$

Let  $B = 2 \max\{|\xi| : \xi \in \Sigma_\rho\}$ . By an integration by parts in  $\rho$  (after interchanging the order of integration in (37)) we obtain

$$(39) \quad |\langle S_{x,t}, S_{\tilde{x},\tilde{t}} \rangle| \leq C_N 2^{kd} (1 + 2^k |t - \tilde{t}|)^{-N} \quad \text{if } |t - \tilde{t}| \geq B|x - \tilde{x}|,$$

and (35) follows from (38) and (39).

We now prove (36) and in view of (39) we may assume  $|t - \tilde{t}| < B|x - \tilde{x}|$ . We distinguish two cases. In the first case we assume  $|\langle \frac{x-\tilde{x}}{|x-\tilde{x}|}, \mathbf{n}_o \rangle| \leq 1 - \varepsilon$ . We note that  $|\Xi(u) - \Xi(u_o)| = O(\varepsilon^2)$  and  $|(\nabla_u \langle x - \tilde{x}, \Xi(u) \rangle)_{u=u_o}| \sim |\Pi_{\mathbf{n}_o^\perp}(x - \tilde{x})|$ . Hence we have  $|\nabla_u \langle x - \tilde{x}, \Xi(u) \rangle| \geq c\varepsilon|x - \tilde{x}|$  on the support of  $b$  provided that  $\varepsilon$  is sufficiently small, and higher derivatives of  $\langle x - \tilde{x}, \Xi(u) \rangle$  are  $O(|x - \tilde{x}|)$ . Thus, integrating by parts in the inner  $u$ -integral in (37), we get (36). We now consider the second case  $|\langle \frac{x-\tilde{x}}{|x-\tilde{x}|}, \mathbf{n}_o \rangle| \geq 1 - \varepsilon$ . This means that  $\frac{x-\tilde{x}}{|x-\tilde{x}|} = s\mathbf{n}_o + O(\varepsilon)$  where  $s = 1$  or  $s = -1$ . From the condition on  $(x - \tilde{x}, t - \tilde{t})$  in (36) we see that the  $\rho$ -derivative of the phase is

$$\begin{aligned} \langle x - \tilde{x}, \Xi(u) \rangle + t - \tilde{t} &= \langle x - \tilde{x}, \xi_o \rangle + \frac{\langle x - \tilde{x}, \mathbf{n}_o \rangle}{\langle \xi_o, \mathbf{n}_o \rangle} + O((\varepsilon^2 + \epsilon_o)|x - \tilde{x}|) \\ &= s|x - \tilde{x}| \left( \langle \xi_o, \mathbf{n}_o \rangle + \frac{1}{\langle \xi_o, \mathbf{n}_o \rangle} \right) + O((\varepsilon + \epsilon_o)|x - \tilde{x}|). \end{aligned}$$

Now  $|\langle \xi_o, \mathbf{n}_o \rangle| \geq c > 0$  which is a consequence of the homogeneity relation  $\rho(\xi) = \langle \xi, \nabla \rho(\xi) \rangle$ . Hence,  $|\langle x - \tilde{x}, \Xi(u) \rangle + t - \tilde{t}| \gtrsim |x - \tilde{x}|$  if  $\varepsilon$  and  $\epsilon_o$  are small enough, and another integration by parts in  $\rho$  gives (36).  $\square$

*Acknowledgement.* The second named author would like to thank Fedya Nazarov.

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SANGHYUK LEE, SCHOOL OF MATHEMATICAL SCIENCES, SEOUL NATIONAL UNIVERSITY, SEOUL 151-742, KOREA

*E-mail address:* `shlee@math.snu.ac.kr`

ANDREAS SEEGER, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, 480 LINCOLN DRIVE, MADISON, WI, 53706, USA

*E-mail address:* `seeger@math.wisc.edu`