Serre's Conjecture Over Imaginary Quadratic Fields

My research is on the connection between Bianchi modular forms and Galois representations of imaginary quadratic fields. In particular, I am investigating whether an analogue Serre's conjecture holds in this setting.

So far I have been able to

- write Magma code to compute the Bianchi modular forms and the Hecke action for *arbitrary* weights over Euclidean imaginary quadratic fields
- prove the nonexistence of irreducible level 1 mod 2 Galois representations of certain imaginary quadratic fields (*this actually agrees with the numerical data I gathered about the Bianchi modular forms as explained in Section 2*)

In what follows, I will first provide a background for the general mathematical audience. Then I will continue with a description of my results and research plan.

1 Background

A *Galois representation* of a number field K is a continuous representation of $G_K = \mathrm{Gal}(\overline{K}/K)$, the absolute Galois group of K. Let λ be a place of K over the rational prime p and fix an embedding of $\overline{K} \subset \overline{K}_\lambda$ where K_λ is the completion of K at λ. The elements of G_K which can be continuously extended to $\overline{K}_\lambda$ form a subgroup D_λ of G_K. There is a short exact sequence $0 \to I_\lambda \to D_\lambda \to \mathrm{Gal}(\overline{\mathbb{F}}_\lambda/\mathbb{F}_\lambda)$ where $\mathbb{F}_\lambda$ is a finite field of order $q = p^m$ for some m. The group $Gal(\overline{\mathbb{F}}_\lambda/\mathbb{F}_\lambda)$ is topologically generated by the *Frobenius automorphism* $\mathrm{Frob}_\lambda : x \mapsto x^q$. A representation ρ of G_K is *unramified* at λ if $\rho(I_\lambda) = \{1\}$. In this case, all preimages of Frob_λ in D_λ will have the same image under ρ. We denote any preimage with Frob_λ as well. A different embedding $\overline{K} \subset \overline{K}_\lambda$ conjugates D_λ by an element of G_K. Therefore, the characteristic polynomial of $\rho(\mathrm{Frob}_\lambda)$ is well defined.

There is an important connection between Galois representations and automorphic forms. The classical example is that of modular forms. These are complex analytic functions on the upper half-plane satisfying a certain kind of functional equation and growth condition. More precisely, given positive integers k, N, a *modular form* of weight k and level N is a holomorphic function f, which satisfies $f(\gamma(z)) = (cz + d)^k f(z)$ for elements $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $SL_2(\mathbb{Z})$ whose reduction modulo N is upper triangular. In particular, $f(z + 1) = f(z)$ so that f has a Fourier expansion of the form $f = \sum_{n\in\mathbb{Z}} a_n q^n$ around $q = 0$ where $q = e^{2\pi iz}$. The growth condition that $a_n = 0$ for $n < 0$. We say that f is *cuspidal* if it satisfies a certain vanishing property which implies that $a_0 = 0$. The space of modular forms of weight k and level N is finite dimensional and there is a commuting algebra of *Hecke operators* T_n, $n \geq 1$ acting on this space. If $f = \sum_{n=0}^{\infty} a_n q^n$ is a simultaneous eigenvector, called an *eigenform*, then $a_1 \neq 0$ and T_n has eigenvalue a_n/a_1.

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a level N cuspidal eigenform with $a_1 = 1$. Then the field E_f generated by the Fourier coefficients of f is a number field. If λ is a prime of E_f over p and $O_{f,\lambda}$ is the ring of integers of the completion of E_f at λ, then by the work of Eichler, Shimura, Deligne and Serre ([5],[6]), one can *attach* f a Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(O_{f,\lambda})$ in the sense that ρ is unramified at every prime l not dividing pN and if ρ is unramified at the prime l, the characteristic polynomial of $\rho(\mathrm{Frob}_\ell)$ is completely determined by f. In particular, the traces of the Frobenius elements are given by the Fourier coefficients of f, that is $tr(\rho(Frob_l)) = a_l$. Reducing $O_{f,\lambda}$ by its maximal ideal, we get a *mod p representation* $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(k)$ attached to f, where k is a finite field of characteristic p.

In reverse direction, the weak version of Serre's conjecture says that any Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ that is irreducible and odd (i.e. the determinant of the image of a complex conjugation is -1) should arise from a cuspidal eigenform in the manner described in the previous paragraph. The refined version conjectures that there is a cuspidal eigenform with precisely described level and weight that ρ arises from. The refined conjecture [11] has many important consequences, in particular it plays a crucial role in the proof of Fermat's Last Theorem. Today, by the work of Khare and many others, the refined conjecture is proved.

A natural question is whether an analogue of Serre's conjecture holds for Galois representations of other number fields. Buzzard, Diamond and Jarvis [2] formulated a similar conjecture in the case of totally real number fields. In this setting, the role of modular forms is played by Hilbert modular forms. A key fact is that both modular forms and Hilbert modular forms are intimately related to the so called Shimura varieties. This allows algebraic geometric tools to come into play in the study of the conjecture in both settings.

My research focuses on the case of imaginary quadratic fields, which is much less understood. For example, there is no obvious link to algebraic geometry as in the other two cases mentioned above. Here, the corresponding modular forms (the so called Bianchi modular forms) are not as concrete as the classical modular forms are. Nevertheless, they can be realized in certain cohomology spaces which are more tangible. Let K be an imaginary quadratic field and O_K be its ring of integers. A finite index subgroup Γ of $SL_2(O_K)$ is called a *Bianchi subgroup*. A torsion free Bianchi subgroup Γ acts freely and discontinuously on the three dimensional hyperbolic space $\mathcal{H} = \{(z, r) |\ z \in \mathbb{C}, r \in \mathbb{R}^+\}$ via hyperbolic isometries. The quotient space $\Gamma \backslash \mathcal{H}$ is a contractible , smooth 3-manifold. The cohomology group $H^1(\Gamma \backslash \mathcal{H}, \mathbb{C})$ has a commuting algebra of Hecke operators acting on them. The classes of this Hecke module can be identified with Bianchi modular forms. For K of class number one, Grunewald, Helling and Mennicke [9] and later Cremona [4] made explicit calculations of Hecke eigenclasses of $H^1(\Gamma\backslash\mathcal{H},\mathbb{C})$ and related them to elliptic curves over K. In the direction of Serre's conjecture, Elstrodt, Grunewald and Mennicke [7] and later Figueiredo [8] considered the mod p cohomology $H^1(\Gamma\backslash\mathcal{H},\mathbb{F}_p)$ and related the Hecke eigenclasses to mod p Galois representations of K of class number one, by showing that the traces of the images of Frobenius elements matched the eigenvalues of the Hecke eigenclass for many primes. Both methods considered the dual homology group and used modular symbols to compute the homology.

2 My Contributions

I wrote a Magma program to compute arbitrary weight Bianchi modular forms. The natural generalization of the notion of weight is an irreducible $\mathrm{GL}_2(\mathbb{F}_p)$-module. Grunewald and Figueiredo considered the cohomology group $H^1(\Gamma\backslash\mathcal{H},\mathbb{F}_p)$ which gives only "trivial weight" Bianchi modular forms. To be able to formulate a full scale Serre's conjecture, we need to consider all the possible weights. As $\Gamma\backslash\mathcal{H}$ is a contractible space, the cohomology of $\Gamma\backslash\mathcal{H}$ is isomorphic to the group cohomology of Γ. As part of an ongoing project with Grunewald, I wrote Magma code to compute $H^1(PSL_2(O_K),E_{k,l}(\mathbb{F}_p))$ and the associated Hecke action for $K=\mathbb{Q}(\sqrt{d})$ with $d=-1,-2$, where the coefficient module $E_{k,l}(\mathbb{F}_p)$ is the tensor product of the k^{th} symmetric power of $\mathbb{F}_p^2$ with the natural Γ action and the l^{th} symmetric power of $\mathbb{F}_p^2$ with the Γ action twisted by complex conjugation. I am currently adapting the program for arbitrary Γ. This program is the main tool for testing the conjecture over Euclidean imaginary quadratic fields.

In [12], I proved the following theorem which is related to Serre's conjecture over both real and imaginary quadratic fields.

Theorem. There is no irreducible Galois representation $G_K\to\mathrm{GL}_2(\overline{\mathbb{F}}_2)$ that is unramified away from $\{2,\infty\}$ for $K=\mathbb{Q}(\sqrt{d})$ with $d=6,5,3,2,-1,-2,-3,-5,-6$.

The analogue of the weak version of Serre's conjecture says that such a Galois representation is supposed to come from a Hecke eigenclass in some $H^1(PSL_2(O_K),E_{k,l}(\mathbb{F}_2))$. The eigenvalue systems for these cohomology groups that I computed with the program were trivial for all the weights and Hecke operators I considered. So if the eigenvalue systems are indeed trivial as the data suggests, then the conjecture implies that there is no such representation as asserted by the theorem.

As an interesting byproduct of the theorem, I reprove the following fact.

Corollary. There is no elliptic curve with good reduction everywhere over the fields $K = \mathbb{Q}(\sqrt{d})$ with $d = 5, 3, 2, -1, -2, -3, -5, -6$.

3 My Research Plan

Here are some problems that I am currently working on and some that I am planning to start soon.

1. I will formulate a refined version of the conjecture. The nontrivial part is specify the set of weights attached to the mod p representation. I expect that this set will be very similar to the one suggested by Buzzard, Diamond and Jarvis.

2. Currently, I am working on adapting my Magma program to higher level computations. This should be ready very soon. The algorithm works only for the Euclidean imaginary quadratic fields. I want to explore the methods of Cremona[4] and Bygott [3], Ash, Gunnells and McConnell [10] to find ways of working with non-Euclidean imaginary quadratic fields too. Another issue that I want to work on is the efficiency of the code.

3. An important task is producing Galois representations of imaginary quadratic fields with levels and weights suitable for the computation of the corresponding cohomology spaces. It boils down to finding extensions with prescribed ramification and Galois group. Aside from systematically searching databases of number fields, I also started to search for elliptic curves over imaginary quadratic fields with certain reduction properties hoping to get level 1 representations coming from their torsion points. Next, I want to investigate liftings of projective representations, preferrably ones with nonsolvable images.

4. A very interesting question is to understand how the "classes that do not lift" fit into the picture. In the classical case, the mod p modular forms are the reductions of the characteristic 0 ones. This is not true for the imaginary quadratic case. By a theorem of Taylor [13], the mod p classes that lift to characteristic 0 have matching Galois representations. It is interesting to test whether there are Galois representations matching the classes that do not lift. I will start identifying these classes and look for matching representations as soon as I improve my Magma program for higher levels.

5. Another problem I will consider is the investigation of the congruences between higher and lower weight Bianchi modular forms as in Ash and Stevens [1].

4 References

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