Math 521 Advanced Calculus, Lecture 2, Spring 2003 Extra Problem Set

1. Binomial series.

(a) First show the following converse to the ratio test: if $c_n \neq 0$ for all n and

$$\varliminf_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| > 1$$

then $\sum c_n$ cannot converge.

(b) Assume s is a real other than a nonnegative integer. Show that the binomial series

$$\sum_{n=0}^{\infty} \binom{s}{n} x^n$$

has radius of convergence at most 1.

(c) Use the ratio test to show that the binomial series has radius of convergence at least 1.

2. We discovered that a function represented by a power series with positive radius of convergence is infinitely differentiable. It is obviously relevant to find out whether the class of power series coincides with the class of infinitely differentiable functions. This exercise resolves the issue. Define

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

(a) Show that $f \in C^\infty(\mathbf{R})$ [this means that f can be differentiated arbitrarily many times at every point of $\mathbf{R}$] and $f^{(n)}(0) = 0$.

Hints: You need to check differentiability at 0 separately, by appeal to the definition of the derivative. Start by proving inductively that for every n, there exists some polynomial p such that

$$f^{(n)}(x) = p\left(\frac{1}{x}\right) e^{-\frac{1}{x}} \qquad \text{for } x > 0.$$

(b) Show that there does not exist a power series $\sum a_n x^n$ such that $f(x) = \sum a_n x^n$ in some nonempty interval $(-R, R)$.

3. Show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

converges iff $p > 1$. (*Hint:* Integral test.) Why is the series starting at $n = 2$ instead of $n = 1$?

4. Find the function $f(x)$ whose power series is $\sum_{n=1}^{\infty} nx^n$. *Hint:* Differentiate a geometric series.

5. Let $f(x) = \cos(\pi e^x)$. Find

$$\lim_{x \to 0} \frac{f(x) + 1}{x^2}.$$