Due Monday, May 8, 4 PM, in the instructor’s office VV419. Absolutely no collaboration permitted. If you need to discuss the exam, see the instructor. 30 pts total.

Some definitions and notation. For a set $B$ on the real line, $B - x = \{y - x : y \in B\}$. $\mathcal{B}_\mathbb{R}$ denotes the Borel $\sigma$-algebra on $\mathbb{R}$. If $Z$ is a real-valued random variable on $(\Omega, \mathcal{F}, P)$, the distribution $\mu$ of $Z$ is the probability measure $\mu(B) = P\{Z \in B\}$, $B \in \mathcal{B}_\mathbb{R}$.

1. Let $\mu, \nu$ be Borel probability measures on $\mathbb{R}$.
   (a) (4 pts) Let $B \in \mathcal{B}_\mathbb{R}$. Show that the function $g(x) = \mu(B - x)$ is Borel measurable. (Hint: Reasoning via a product space might be helpful. Start by writing $\mu(B - x)$ as an integral. Notice that $y \in B - x$ iff $x + y \in B$.)
   (b) (4 pts) Show that $\alpha(B) = \int_{\mathbb{R}} \mu(B - x) \nu(dx)$ defines a probability measure $\alpha$ on the Borel $\sigma$-algebra of $\mathbb{R}$. The measure $\alpha$ is called the convolution of $\mu$ and $\nu$, and denoted by $\mu * \nu$. (Part (a) assures us that the integration makes sense. Here one needs to check the properties of a probability measure.)
   (c) (3 pts) Show that $\mu * \nu = \nu * \mu$.
   (d) (4 pts) Let $X$ and $Y$ be two independent real-valued random variables on $(\Omega, \mathcal{F}, P)$ such that $X$ has distribution $\mu$ and $Y$ has distribution $\nu$. Show that the random variable $X + Y$ has distribution $\mu * \nu$.

2. (5 pts) Suppose $\{X_n\}$, $X$ are real-valued random variables such that $X_n \to X$ in probability. (Means the same as $X_n \to X$ in measure.) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then also $f(X_n) \to f(X)$ in probability. (Hint: recall the connection with almost everywhere convergence.)
   Give an example of a function $f$ and random variables $\{X_n\}$, $X$ for which the convergence $f(X_n) \to f(X)$ (in prob) fails even though $X_n \to X$ (in prob).
3. (5 pts) For \( i = 1, 2 \) let \( \nu_i \) and \( \mu_i \) be \( \sigma \)-finite positive measures on the measurable space \((X_i, \mathcal{M}_i)\). Assume \( \nu_i \ll \mu_i \) with Radon-Nikodym derivative \( f_i(x_i) = \frac{d\nu_i}{d\mu_i}(x_i) \). Show that then \( \nu_1 \otimes \nu_2 \ll \mu_1 \otimes \mu_2 \) and

\[
\frac{d(\nu_1 \otimes \nu_2)}{d(\mu_1 \otimes \mu_2)}(x_1, x_2) = f_1(x_1)f_2(x_2).
\]

4. (5 pts) Let \((X, \mathcal{M}, \mu)\) be a measure space, \( \mu \) a positive measure. Suppose \( f_n : X \to [-\infty, \infty] \) and \( g : X \to [0, \infty] \) are measurable, \( \int g \, d\mu < \infty \), and \( f_n \geq -g \) for all \( n \). Show that

\[
\int (\lim \inf f_n) \, d\mu \leq \lim \inf \int f_n \, d\mu.
\]

(In other words, the conclusion of Fatou’s Lemma is valid.) Be sure to justify cancellations properly.

Give an example of a sequence of real-valued functions \( \{f_n\} \) for which the conclusion is not valid.