1. Let $0 < \rho < 1$. Define a Borel probability measure $\mu$ on $\mathbb{R}$ by
\[
\mu(B) = \frac{1}{2} \int_\mathbb{R} 1_B(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx + \frac{1}{2} \sum_{k \in \mathbb{N}} \rho(1 - \rho)^{k-1} 1_B(k), \quad B \in \mathcal{B}_\mathbb{R},
\]
and let $X$ be a random variable with distribution $\mu$. The $dx$ integral above stands for an integral with Lebesgue measure. (Probabilistically speaking, $X$ follows a standard normal distribution with probability $1/2$, and a geometric distribution with parameter $\rho$ with probability $1/2$.)

(a) Show that for all $f \in L^1(\mu)$,
\[
\int_\mathbb{R} f \, d\mu = \frac{1}{2} \int_\mathbb{R} f(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx + \frac{1}{2} \sum_{k \in \mathbb{N}} \rho(1 - \rho)^{k-1} f(k).
\]
Hint: start with $f = 1_B$.

(b) Show that $X$ is integrable. (Integrability means $E|X| < \infty$.)

(c) Compute the mean $E(X)$.
Justify all the steps you take rigorously.

2. Exercise 19 on page 59. Uniform convergence means this: $f_n \to f$ uniformly if for every $\varepsilon > 0$ there exists $N < \infty$ such that for all $n \geq N$,
\[
\sup_{x \in X} |f_n(x) - f(x)| \leq \varepsilon.
\]
In other words, the point is that the same $N$ works for all $x$ once $\varepsilon$ is given. In pointwise convergence different $x$ might need a different $N$. So quite obviously uniform convergence implies pointwise convergence.

3. Exercise 48 on page 69. What is the significance of this exercise to Fubini’s theorem?