Math 635 Introduction to Stochastic Calculus, Spring 2003
Take-Home Midterm Exam, Part 2

Due Thursday, March 10

1. *Cross variations.* Recall that for two processes X_t and Y_t,

$$\langle X, Y \rangle_t = \lim_{\|\Delta\| \to 0} \sum (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$$

if the limit in probability exists.

(a) Find $\langle B, W \rangle_t$ for two independent standard Brownian motions B_t and W_t.

(b) Let $Y_t(\omega) = \int_0^t a(\omega, s)\, ds$ for a bounded, adapted process $a(\omega, s)$. Find $\langle B, Y \rangle_t$.

Hint: In both (a) and (b), showing L^2 convergence is convenient.

2. (a) Exercise 8.2 from our text.

(b) Show that $M_t = t^2 B_t - 2\int_0^t sB_s\, ds$ is a martingale.

(c) Let $X_t = B_t^2 - t$. Express X_t as a stochastic integral. Use Theorem 8.6 to find the quadratic variation $\langle X \rangle_t$. We know $M_t = X_t^2 - \langle X \rangle_t$ is a standard process because both terms are. On the other hand, we also know that M_t is a martingale. Hence M_t must be a stochastic integral. Find the expression of M_t as a stochastic integral.

3. Let $B_1(t), B_2(t), B_3(t), \ldots$ be independent standard Brownian motions. Find a process of the form

$$A_t = \int_0^t g(B_1(s), B_2(s))\, ds$$

such that $B_1^2(t)B_2^2(t) - A_t$ is a martingale. How does this generalize to $B_1^2(t)B_2^2(t)B_3^2(t)$?

4. Find a solution Y_t to the SDE

$$dY_t = r\,dt + \alpha Y_t\,dB_t$$

with a given initial value Y_0 independent of the Brownian motion. Check your solution by using Ito's formula to compute dY_t.

($dY_t = r\,dt$ alone would be linear growth, so the equation represents linear growth with a rate that is randomly perturbed proportionally to the size of the current amount Y_t.)