Due 3 PM on Monday, February 3

1. Give an example of a sequence of random variables $X_n$ and a random variable $X$, all defined on the same probability space $(\Omega, \mathcal{F}, P)$, and such that $X_n(\omega) \to X(\omega)$ for each $\omega \in \Omega$ but $EX_n \to EX$ fails. (Limits as $n \to \infty$.) Note that you need to find an example that violates the hypotheses of DCT and MCT (p. 278 in the book).

2. Let $\Omega = \mathbb{R}$ and on this $\Omega$ put the probability measure $P$ defined for Borel sets $A \subseteq \mathbb{R}$ by

$$
P(A) = \frac{\lambda(A \cap [-10, 10])}{20}
$$

where $\lambda$ is Lebesgue measure on $\mathbb{R}$. In words, $P$ is the uniform probability measure on the interval $[-10, 10]$. Examples of values of $P$ are $P([a, b]) = (b - a)/20$ for $[a, b] \subseteq [-10, 10]$, $P([-100, 0]) = 1/2$, and $P(A) = 0$ if $A$ lies outside $[-10, 10]$.

Define the random variable (measurable function) $X : \Omega \to \mathbb{R}$ by

$$
X(\omega) = \begin{cases} 
-5 & \text{if } \omega \leq -5 \\
\omega & \text{if } -5 < \omega \leq 0 \\
k & \text{if } \omega \in (k - 1, k] \text{ for a positive integer } k.
\end{cases}
$$

Note that $X$ is not purely discrete, nor does it have a density.

(a) Find the probabilities $P(X \leq -7)$, $P(X \leq -4)$, $P(X \leq 12)$, and $P(-2 \leq X \leq 3.5)$.

(b) Compute the expectation $E(X)$. Hint. Evaluate the Lebesgue integral $E(X) = \int_{\Omega} X(\omega) P(d\omega)$ piece by piece. Use some common sense and some calculus.

(c) Define the subsets $A = \{-8, 0, 1, 2\}$ and $B = (5, 12]$ of $\Omega$. Decide whether $A$ and $B$ are members of the $\sigma$-algebra $\sigma(X)$ on $\Omega$ generated by $X$. Is there a Borel subset of $(-5, 0]$ that is not a member of $\sigma(X)$? (Note that part (b) does not involve the measure $P$ at all, only properties of $X$ as a function.)
3. (Exercise 4.1(a) in the book: tower property) Use the definition of conditional expectation to prove that if $\mathcal{H}$ is a sub-σ field of $\mathcal{G}$ then
\[
E[E(X|\mathcal{G})|\mathcal{H}] = E(X|\mathcal{H}).
\]

4. Suppose $X$ has $\text{Exp}(\lambda)$ distribution. This is the exponential distribution with parameter $\lambda$, which means that $X$ has density $f(x) = \lambda e^{-\lambda x}$ on $[0, \infty)$, and $f(x) = 0$ on $(-\infty, 0)$.

A possible way to realize a probability space $(\Omega, \mathcal{F}, P)$ for $X$ is to take $\Omega = \mathbb{R}$, $\mathcal{F} = \mathcal{B}_{\mathbb{R}}$, and let the measure $P$ be defined for Borel sets $B$ by
\[
P(B) = \int 1_B(x) f(x) \, dx
\]
where the integral is interpreted as a Lebesgue integral over $\mathbb{R}$. For intervals this gives the familiar values $P((a, b]) = F(b) - F(a)$ where $F$ is the c.d.f. defined by
\[
F(x) = \int_{-\infty}^{x} f(y) \, dy.
\]
On this $\Omega$, define $X(\omega) = \omega$. On this same probability space define a random variable $Y$ by
\[
Y(\omega) = \begin{cases} 3, & \omega \leq 10, \\ 27, & \omega > 10. \end{cases}
\]
Find the random variable $E(X|Y)(\omega)$.

5. Suppose $(X, Y)$ is a pair of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Suppose $(X, Y)$ has a joint density $f(x, y)$. This means that $f$ is a nonnegative function on $\mathbb{R}^2$ that satisfies
\[
E[H(X, Y)] = \int_{\mathbb{R}^2} H(x, y) f(x, y) \, dx \, dy
\]
for any bounded Borel measurable function $H : \mathbb{R}^2 \to \mathbb{R}$. The marginal density $f_Y$ of $Y$ is defined by
\[
f_Y(y) = \int_{\mathbb{R}} f(x, y) \, dx.
In an elementary probability course we define the conditional density of $X$, given that $Y = y$, by

$$f(x|y) = \begin{cases} \frac{f(x, y)}{f_Y(y)} & \text{if } f_Y(y) > 0 \\ 0 & \text{if } f_Y(y) = 0. \end{cases}$$

Let $\phi$ be a bounded Borel function on $\mathbb{R}$ and define

$$Z(\omega) = \int_{\mathbb{R}} \phi(x) f(x|Y(\omega)) \, dx.$$

Show that $Z$ is the conditional expectation $E[\phi(X)|Y]$. The measurability issue is immediate as $Z$ is a function of $Y$. For the other part of the definition of conditional expectation, you need to check that

$$E[Z \psi(Y)] = E[\phi(X) \psi(Y)]$$

for an arbitrary bounded Borel function $\psi$. To check this, evaluate the expectations by integrating over $\mathbb{R}^2$ with the help of the densities.