Math 635 Introduction to Stochastic Calculus, Spring 2014
Homework 3

Due 3 PM on Wednesday, Feb. 26.

**Generalities.** Throughout these exercises \(B_t = \{B_t : t \in \mathbb{R}_+\}\) is standard Brownian motion.

Recall that if \(X \sim \text{Poisson}(\alpha)\) for \(0 < \alpha < \infty\) then \(X\) has probability mass function \(P(X = k) = e^{-\alpha} \alpha^k / k!\) for \(k \in \mathbb{Z}_+\).

The strong law of large numbers (SLLN) states the following. If \(\{X_k\}_{k \in \mathbb{N}}\) are i.i.d. (independent and identically distributed) random variables with \(E|X_1| < \infty\) and \(S_n = X_1 + \cdots + X_n\), then \(n^{-1}S_n \to EX_1\) almost surely as \(n \to \infty\).

1. Exercise 2.4 from page 28 in the book.

2. In this exercise you prove that \(t^{-1}B_t \to 0\) almost surely as \(t \to \infty\).
   (a) Apply the SLLN to show that \(n^{-1}B_n \to 0\) almost surely as \(n \to \infty\) along positive integers.
   (b) Use Doob’s maximal inequality and the Borel-Cantelli lemma to prove that
   \[
   \lim_{N \to \infty} \frac{1}{n} \sup_{t \in [n, n+1]} |B_t - B_n| = 0 \quad \text{a.s.}
   \]
   (c) Combine (a)–(b) to prove that \(t^{-1}B_t \to 0\) almost surely as \(t \to \infty\).

3. Define the process
   
   \[
   Y_t = \begin{cases} 
   0 & \text{if } t = 0 \\
   tB_{1/t} & \text{if } t > 0.
   \end{cases}
   \]

   This is called the **time inversion of Brownian motion**. Show that \(Y_t\) is a standard Brownian motion. **Hint.** You will need the result of the previous exercise.
4. The cross variation of two processes $X_t$ and $Y_t$ is defined as

$$
\langle X, Y \rangle_t = \lim_{\|\Delta\| \to 0} \sum_i (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})
$$

if the limit in probability exists, where $\Delta = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ is a partition of $[0, t]$ with mesh $\Delta = \max_i (t_i - t_{i-1})$. Find $\langle B, W \rangle_t$ for two independent standard Brownian motions $B$ and $W$. (Calculate the $L^2$ limit, as we did in class for the quadratic variation of Brownian motion.)

5. In addition to Brownian motion, another stochastic process of central importance is the Poisson process. Let $0 < \lambda < \infty$. Let the process $\{N_t\}_{t \in \mathbb{R}_+}$ be adapted to the filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ (that is, for each $t$ the random variable $N_t$ is $\mathcal{F}_t$-measurable). Then $\{N_t\}$ is a rate $\lambda$ Poisson process relative to the filtration $\{\mathcal{F}_t\}$ if the following conditions are satisfied.

(i) $N_0 = 0$.

(ii) For each $t > s$, the increment $N_t - N_s$ is Poisson($\lambda(t-s)$) distributed and independent of $\mathcal{F}_s$.

(iii) The paths $t \mapsto N_t$ are right-continuous.

Condition (ii) implies in particular that increments $\{N_{t_k} - N_{t_{k-1}} : 1 \leq k \leq n\}$ for $0 = t_0 < t_1 < \cdots < t_n$ are independent. A Poisson process is piecewise constant and jumps one step up at a time, with exponential waiting times between jumps.

(a) Show that the process $X_t = N_t - \lambda t$ is a martingale with respect to the filtration $\mathcal{F}_t$.

(b) Recall that for Brownian motion $B_t^2$ is a submartingale and $B_t^2 - t$ is a martingale. Find the quantity $a_t$ such that $(N_t - \lambda t)^2 - a_t$ is a martingale, and of course show that you have found a martingale. (These are continuous-time examples of the Doob decomposition of Exercise 2.4 p. 28.)