Due 3 PM on Monday, March 24. Note: if you do not intend to work during spring break, you need to do the problems this week!

**Generalities.** Throughout these exercises \( B_t = \{ B_t : t \in \mathbb{R}_+ \} \) is standard Brownian motion.

1. Let \( f \in \mathcal{H}^2[0,T] \) and \( \tau \) be a stopping time. Show that for \( 0 \leq t \leq T \),
\[
\int_0^{\tau(t) \wedge t} f(\omega, s) dB_s(\omega) = \int_0^t f(\omega, s) 1\{s \leq \tau(\omega)\} dB_s(\omega) \quad \text{a.s.}
\]

You can follow the outline below or produce your own proof using facts in the textbook.

(i) Let \( \tau_n = 2^{-n}([2^n \tau] + 1) \). Check that \( \tau_n \) is a stopping time and \( \tau_n \downarrow \tau \) as \( n \to \infty \). Check that \( \tau \geq k/2^n \text{ iff } \tau_n \geq (k+1)/2^n \).

(ii) Let \( \ell(n) = [2^n T] \). Start with the telescoping sum
\[
\int_0^{\tau_n \wedge t} f dB = \sum_{k=0}^{\ell(n)} 1\{\tau_n \geq \frac{k+1}{2^n}\} \left( \int_0^{\frac{k+1}{2^n} \wedge t} f dB - \int_0^{\frac{k}{2^n} \wedge t} f dB \right) \quad (1)
\]

Use facts we have proved, such as \( \int_0^t f(s) dB_s = \int_0^T 1_{(0,t]}(s) f(s) dB_s \) for \( 0 < t < T \), \( 1_A \int_s^t f dB = \int_s^t 1_A f dB \) for \( s < t \) and \( A \in \mathcal{F}_s \), and linearity, to turn the right-hand side of (1) into \( \int_0^t 1_{(0,\tau_n]}(s) f(s) dB_s \).

(iii) Show that
\[
\int_0^t 1_{(0,\tau_n]}(s) f(s) dB_s \longrightarrow \int_0^t 1_{(0,\tau]}(s) f(s) dB_s \quad \text{in } L^2, \text{ as } n \to \infty.
\]

(iv) Use path continuity on the left side of (1).
2. The integral of a step function in $\mathcal{L}^2_{\text{loc}}[0,T]$. Fix a partition $0 = t_0 < t_1 < \cdots < t_M = T$, and random variables $a_0, \ldots, a_{M-1}$. Assume that $a_i$ is almost surely finite and $\mathcal{F}_{t_i}$-measurable, but make no integrability assumptions on them. Define

$$g(\omega, s) = \sum_{i=0}^{M-1} a_i(\omega) \mathbf{1}_{(t_i, t_{i+1}]}(s).$$

The task is to show that $g \in \mathcal{L}^2_{\text{loc}}$ (virtually immediate) and that

$$\int_0^t g(s) \, dB_s = \sum_{i=0}^{M-1} a_i(B_{t_{i+1} \wedge t} - B_{t_i \wedge t})$$

as we would expect.

**Hint.** Here is one possible localizing sequence:

$$\nu_n(\omega) = \inf \{ t : |g(\omega, t)| \geq n \text{ or } t \geq T \}.$$ 

Show that $g(\omega, t) \mathbf{1}_{\{ t \leq \nu_n(\omega) \}}$ is actually also a step function with the same partition. Then you know exactly what the approximating integral

$$X_{n,t}(\omega) = \int_0^t g(\omega, s) \mathbf{1}_{\{ s \leq \nu_n(\omega) \}} \, dB_s(\omega)$$

looks like, and the definition in Section 7.1 can be applied.

3. The next two problems are together only because both are short problems.

(a) Prove that the random variable

$$X = \int_0^T (B_s + s) \, dB_s$$

is in $L^2$, and compute its mean and variance.

(b) Find an expression for

$$\int_0^t (\cos B_s) \, dB_s$$

that does not involve any stochastic integrals.