Math 635 Introduction to Stochastic Calculus, Spring 2003
Take-Home Final Exam

Instructions: Give solutions to the problems below that are concise, yet as rigorous as possible, given the amount of theory we have covered.

Due Tuesday, May 13, 4 PM in the instructor's office or mailbox

1. (a) One of the basic things we showed was the Ito isometry

$$E\Big[\Big(\int_0^t f(s)\,dB_s\Big)^2\Big] = E\int_0^t f(s)^2\,ds$$

for $f \in \mathcal{H}^2[0,t]$. Use this to derive the identity

$$E\Big[\int_0^t f(s)\,dB_s \cdot \int_0^t g(s)\,dB_s\Big] = E\int_0^t f(s)g(s)\,ds$$

for $f, g \in \mathcal{H}^2[0,t]$.

(b) Let

$$U(t) = \begin{bmatrix} u_{11}(t) & u_{12}(t) \\ u_{21}(t) & u_{22}(t) \end{bmatrix}$$

be a bounded measurable 2×2-matrix valued function on $[0,T]$. Let $\vec{B}_t$ be a standard 2-dimensional Brownian motion, and define

$$\vec{X}_t = \begin{bmatrix} X_t^1 \\ X_t^2 \end{bmatrix} = \int_0^t U(s)\,d\vec{B}_s.$$

Use part (a) to find the covariance matrix of $\vec{X}_t$. Express it in a compact form utilizing the matrix $U(s)$.

(c) Find a (deterministic) function $\tau : [0,\infty) \to [0,\infty)$ such that

$$X_t = \int_0^t e^s\,dB_s \qquad \text{and} \qquad Y_t = B_{\tau(t)}$$

have the same distribution as processes. *Hint:* Proposition 7.6 points in the right direction. This problem is part of Exercise 7.1 in the book, but the time-change results of Ch. 7 are not needed for this.

2. Let B_t and W_t be two independent Brownian motions, and let

$$Y_t = \int_0^t B_s \, dW_s.$$

How is the random variable Y_t distributed? (One way to proceed: Compute the characteristic function of Y_t by a step function approximation. Recall that L^2 convergent random variables converge almost surely along a sufficently fast subsequence, so you can take a limit of characteristic functions.)

3. (a) Exercise 9.2 from p. 150 in our book.

(b) Exercise 9.3 from p. 150 in our book.

4. Let $\mu > 0$ be a constant, and suppose $(Y_t : 0 \le t \le T)$ is an Ornstein-Uhlenbeck process defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. Assume $Y_\cdot$ has initial value $Y_0 = 0$ and mean square

$$E[Y_t^2] = \frac{1 - e^{-2\mu t}}{2\mu}.$$

(a) Express Y_t in the form of a standard process (Def. 8.1, p. 126).

(b) Let A be some Borel subset of the space $C[0,T]$ of continuous functions on $[0,T]$. Let

$$\beta = \tilde{P}(Y_\cdot \in A)$$

be the probability that the path of $Y_\cdot$ lies in the set A. We want to calculate β as an expectation under Wiener measure P on $C[0,T]$. The task is to find a measurable function Ψ on $C[0,T]$ so that

$$\beta = E_P[\mathbf{1}_A \cdot \Psi].$$

Can you express Ψ without stochastic intergrals?