735 Stochastic Analysis Fall 2003 Homework 2

Due Thursday, Oct. 23

Hand in **all** problems. These problems appear as exercises in the notes, in many cases with more hints. Please explain your work well. The grader does not know what has gone on in our lectures. If you do not know enough analysis/measure theory to justify a step that looks plausible to you, *identify the step as such* and carry on with your reasoning.

1. Let B and X be two independent Brownian motions. Following our proof of $[B]_t = t$, show that $[B, X] = 0$ by showing

$$\lim_{\text{mesh}(\pi)\to 0} \sum_{i=0}^{m(\pi)-1} (B_{t_{i+1}} - B_{t_i})(X_{t_{i+1}} - X_{t_i}) = 0 \quad \text{in } L^2(P).$$

2. Let $N = \{N(t) : 0 \le t < \infty\}$ be a homogeneous rate α Poisson process with respect to $\{\mathcal{F}_t\}$. We saw that for the compensated Poisson process $M_t = N_t - \alpha t$, the quadratic variation is $[M]_t = N_t$ while $\langle M \rangle_t = \alpha t$. It follows that N cannot be a natural increasing process. In this exercise you show that the naturalness condition fails for N.

(a) Let $\lambda > 0$. Show that

$$X(t) = \exp\{-\lambda N(t) + \alpha t(1 - e^{-\lambda})\}$$

is a martingale.

(b) Show that N is not a natural increasing process, by showing that for X defined above, the condition

$$E\int_{(0,t]} X(s)dN(s) = E\int_{(0,t]} X(s-)dN(s)$$

fails. (The meaning of the expectation–integral is that first for a fixed ω, the function $s \mapsto X(s, \omega)$ is integrated against the (positive) Lebesgue-Stieltjes measure of the function $s \mapsto N(s, \omega)$. The resulting function of ω is averaged over the probability space.)

The remainder of the homework concerns Chapter 4: Stochastic Integration with respect to Brownian Motion.

3. In the our definition of simple predictable processes we required the ξ_i bounded. Now let

$$X(t) = \sum_{i=1}^{m-1} \eta_i \mathbf{1}_{(s_i, s_{i+1}]}(t)$$

where $0 \le s_1 < \cdots < s_m < \infty$ and each $\eta_i \in L^2(P)$ is $\mathcal{F}_{s_i}$-measurable. Show that $X \in \mathcal{L}_2(B)$ and

$$\int_0^t X(s)\, dB_s = \sum_{i=1}^{m-1} \eta_i (B_{t \wedge s_{i+1}} - B_{t \wedge s_i}).$$

Hint. Check that a sequence of approximating simple processes is given by

$$X_k(t) = \sum_{i=1}^{m-1} \eta_i^{(k)} \mathbf{1}_{(s_i, s_{i+1}]}(t)$$

with truncated variables $\eta_i^{(k)} = (\eta_i \wedge k) \vee (-k)$. See hints in notes.

4. Show that B_t^2 is a process in $\mathcal{L}_2(B)$ and evaluate

$$\int_0^t B_s^2\, dB_s$$

by following the example of $\int_0^t B_s\, dB_s$ in the notes. In other words, do not use a shortcut such as Ito's formula even if you already know it. The purpose here is to compute a nontrivial stochastic integral from basic principles at least once.

Answer: $\frac{1}{3} B_t^3 - \int_0^t B_s\, ds$.

5. [Integral of a step function in $\mathcal{L}(B)$] Fix $0 = t_0 < t_1 < \cdots < t_M < \infty$, and random variables $\eta_0, \ldots, \eta_{M-1}$. Assume that η_i is almost surely finite and $\mathcal{F}_{t_i}$-measurable, but make no integrability assumption. Define

$$g(s,\omega) = \sum_{i=0}^{M-1} \eta_i(\omega)\mathbf{1}_{(t_i,t_{i+1}]}(s).$$

The task is to show that $g \in \mathcal{L}(B)$ (virtually immediate) and that

$$\int_0^t g(s)\,dB_s = \sum_{i=0}^{M-1} \eta_i(B_{t_{i+1}\wedge t} - B_{t_i\wedge t})$$

as one would expect.

Hints. Let

$$\sigma_n(\omega) = \inf\{t : |g(t,\omega)| \geq n\}.$$

(Recall the convention $\inf\emptyset = \infty$.) Show that the σ_n's are stopping times and form a localizing sequence for g. Show that

$$g_n(t,\omega) = g(t,\omega)\mathbf{1}\{t \leq \sigma_n(\omega)\}$$

is also a step function with the same partition. Then we know what the approximating integrals

$$Y_n(t,\omega) = \int_0^t g_n(s,\omega)\,dB_s(\omega)$$

look like.