735 Stochastic Analysis Fall 2003 Homework 3

Due Thursday, Nov. 13

Hand in **all** problems.

1. Let $0 < a < b < \infty$ be constants. If G is a BV function and Λ_G its Lebesgue-Stieltjes measure, then $\Lambda_G(a,b] = G(b) - G(a)$ while $\Lambda[a,b) = G(b-) - G(a-)$.

Let M be a cadlag L^2-martingale. By definition of the stochastic integral of a predictable simple process,

$$\int_{(0,t]} \mathbf{1}_{(a,b]}(s)\, dM_s = M_{b\wedge t} - M_{a\wedge t}.$$

The Problem: Find the stochastic integral

$$\int_{(0,t]} \mathbf{1}_{[a,b)}(s)\, dM_s.$$

The integrand $X_t(\omega) = \mathbf{1}_{[a,b)}(t)$ is a predictable process because it does not depend on ω. You need not show this.

Hint: One way to approach this is to show that $\mathbf{1}_{(a-1/n,b-1/n]}$ converges to $\mathbf{1}_{[a,b)}$ in $\mathcal{L}_2(M,\mathcal{P})$.

2. *Computations with the Poisson process.* Let N be a homogeneous rate α Poisson process, and $M_t = N_t - \alpha t$ the compensated Poisson process, which we showed to be an L^2-martingale. Let us employ the notation $0 < \tau_1 < \tau_2 < \ldots < \tau_{N(t)}$ for the jump times of N in $(0,t]$.

The path $t \mapsto M_t$ is BV, hence our theory implies that the stochastic integral

$$\int_{(0,t]} X(s)\, dM(s)$$

can be evaluated as a pathwise Lebesgue-Stieltjes integral when the integrand is a predictable process for which there exist stopping times $\sigma_k \nearrow \infty$ such that $\mathbf{1}_{(0,\tau_k]}X$ is a bounded process, for each k.

Among other things, this exercise will help us conclude that N is not a predictable process.

(a) Show that $M_t^2 - \alpha t$ is a martingale, by verifying the martingale property with the help of the independence of $N(t) - N(s)$ and $\mathcal{F}_s$.

(b) Show that

$$E\left[\sum_{i=1}^{N(t)} \tau_i\right] = \frac{\alpha t^2}{2}.$$

Hint: For a homogeneous Poisson process, given that there are n jumps in an interval I, the locations of the jumps are n i.i.d. uniform random variables in I.

(c) Compute the integral

$$\int_{(0,t]} N(s-)\, dM(s).$$

"Compute" means to find a reasonably simple formula in terms of N_t and the τ_i's. The Lebesgue-Stieltjes approach is one possibility, but not the only one. $N(s-)$ is predictable, since it is left-continuous.

(d) Use the formula you obtained in part (c) to check "by hand" that the process $\int N(s-)\, dM(s)$ is a martingale. (Of course, our theory gives this but the point here is to obtain the conclusion through computation. Parts (a) and (b) take care of parts of the work.)

(e) Suppose N were predictable. Then the stochastic integral $\int N\, dM$ would exist and would be a martingale. Show that this does not give a martingale and conclude that N cannot be predictable.

Hints: Rather than redo a lot of work, it might be easiest to find

$$\int_{(0,t]} N(s)\, dM(s) - \int_{(0,t]} N(s-)\, dM(s) = \int_{(0,t]} \big(N(s) - N(s-)\big)\, dM(s)$$

via the Lebesgue-Stieltjes approach, and use the fact that the integral of $N(s-)$ *is* a martingale.

3. Let M and N be cadlag local L^2-martingales and $X \in \mathcal{L}(M, \mathcal{P})$ and $Y \in \mathcal{L}(N, \mathcal{P})$ admissible integrands, all defined on the same probability space with the same filtration. Prove this statement:

Lemma. Let σ be a stopping time, and suppose $X_t(\omega) = Y_t(\omega)$ and $M_t(\omega) = N_t(\omega)$ for all $t \leq \sigma(\omega)$. Show that $(X \cdot M)_{t \wedge \sigma} = (Y \cdot N)_{t \wedge \sigma}$ for all $0 \leq t < \infty$.

Note: You may assume the statement for the case where $M, N \in \mathcal{M}_2$, $X \in \mathcal{L}_2(M, \mathcal{P})$ and $Y \in \mathcal{L}_2(N, \mathcal{P})$. In other words, you need only provide the localization argument.