735 Stochastic Analysis Fall 2003 Take-Home Final

No collaboration permitted. Hand in **all** problems. **Due by noon on Monday, Dec. 15.** You are welcome to hand in earlier.

1. (a) Let B be standard one-dimensional Brownian motion. Show that for a continuously differentiable nonrandom function ϕ,

$$\int_0^t \phi(s)\,dB_s = \phi(t)B_t - \int_0^t B_s \phi'(s)\,ds.$$

(b) Define

$$V_1(t) = \int_0^t B_s\,ds, \quad V_2(t) = \int_0^t V_1(s)\,ds,$$

and generally

$$V_n(t) = \int_0^t V_{n-1}(s)\,ds = \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n\, B_{s_n}.$$

V_n is known as n times integrated Brownian motion, and appears in applications in statistics.

Show that

$$V_n(t) = \frac{1}{n!}\int_0^t (t-s)^n\,dB_s.$$

Then show that the process

$$M_n(t) = V_n(t) - \sum_{j=1}^{n} \frac{t^j}{j!(n-j)!}\int_0^t (-s)^{n-j}\,dB_s$$

is a martingale.

2. Let X and Y be independent rate α Poisson processes.

(a) Find the covariation $[X,Y]$.

(b) Find a process U_t such that $X_tY_t - U_t$ is a martingale.

3. Let B_t be Brownian motion in $\mathbf{R}^k$ started at point $\mathbf{z} \in \mathbf{R}^k$. As before, let

$$\sigma_R = \inf\{t \geq 0 : |B_t| \geq R\}$$

be the first time the Brownian motion leaves the ball of radius R. Compute the expectation $E^{\mathbf{z}}[\sigma_R]$ as a function of $\mathbf{z}$, k and R.

Hints and comments. You may take for granted that $P^{\mathbf{z}}(\sigma_R < \infty) = 1$. Give rigorous justification for limits. Start by applying Itô's formula to $f(\mathbf{x}) = x_1^2 + \cdots + x_k^2$.

4. Let B be standard one-dimensional Brownian motion. Find a solution Y_t to the SDE

$$dY_t = r\,dt + \alpha Y_t\,dB_t$$

with a given initial value Y_0 independent of the Brownian motion.

Hint. You might try a suitable integrating factor to guess at a solution. Then check your solution by using Itô's formula to compute dY_t.

Comment. $dY_t = r\,dt$ alone would be linear growth, so the equation represents linear growth with a rate that is randomly perturbed proportionally to the size of the current amount Y_t.

5. (a) Let Y be a cadlag semimartingale. Find $[[Y]]$ (the quadratic variation of the quadratic variation) and the covariation $[Y, [Y]]$.

(b) Let Y be a continuous semimartingale. Show that

$$X_t = X_0 \exp\left\{\alpha t + \beta Y_t - \tfrac{1}{2}\beta^2 [Y]_t\right\}$$

solves the SDE

$$dX = \alpha X\,dt + \beta X\,dY.$$