HOMEWORK 3 SOLUTIONS

Problem 1

A basic property of Poisson random variables is that if $\{X_k\}$ are i.i.d. Poisson(θ), then S_n is Poisson($n\theta$). Thus,

$$\sum_{0\leq k\leq na}\frac{e^{-n\theta}(n\theta)^k}{k!}=\mathbb{P}(S_n\leq na)=\mathbb{P}(S_n-n\theta\leq n(a-\theta)).$$

By the WLLN, for all $\epsilon>0$

$$\mathbb{P}\left(\left|\frac{S_n}{n}-\theta\right|\geq\epsilon\right)\to 0.$$

Hence, if $a<\theta$ we have

$$\mathbb{P}(S_n-n\theta\leq n(a-\theta))\leq\mathbb{P}(|S_n-n\theta|\geq n(\theta-a))\to 0,$$

while for $a>\theta$,

$$\mathbb{P}(S_n-n\theta\leq n(a-\theta))=1-\mathbb{P}(S_n-n\theta>n(a-\theta))\to 1.$$

Problem 2

$$\begin{aligned}\mathbb{P}\left(S_n\leq\frac{\mathbb{E}S_n}{2}\right)&\leq\mathbb{P}\left(|S_n-\mathbb{E}S_n|\geq\frac{\mathbb{E}S_n}{2}\right)\leq\frac{4Var(S_n)}{(\mathbb{E}(S_n))^2}\\&=\frac{4nVar(X_{n1})}{(\mathbb{E}(S_n))^2}\leq\frac{4nVar(X_{n1}^2)}{(\mathbb{E}(S_n))^2}\\&\leq\frac{4Cn\mathbb{E}(X_{n1})}{(\mathbb{E}(S_n))^2}=\frac{4C\mathbb{E}(S_n)}{(\mathbb{E}(S_n))^2}=\frac{4C}{\mathbb{E}S_n}\to 0.\end{aligned}$$

Consequently, if n is large enough so that $\mathbb{E}S_n>2k$, then

$$\mathbb{P}(S_n\leq k)\leq\mathbb{P}\left(S_n\leq\frac{\mathbb{E}S_n}{2}\right)\to 0$$

by the above.

Problem 3

(a)*Claim 1:* $X_n\to 0$ a.s. $\Leftrightarrow\sum p_k<\infty$.

($\Leftarrow$) The Borel - Cantelli lemma implies $\mathbb{P}(X_n=1\quad i.o.)=0$, therefore

$$\mathbb{P}(\exists n_0(\omega)\text{ s.t. }\forall n\geq n_0\Rightarrow X_n=0)=1.$$

($\Leftarrow$) $\sum p_k=\infty\Rightarrow\mathbb{P}(X_n=1\quad i.o.)=1$ by the second B-C lemma, hence with probability 1, $X_n\to 0$ fails.

Claim 2: $X_n \xrightarrow{P} 0 \Leftrightarrow p_n \to 0$.

$$\begin{aligned} X_n \xrightarrow{P} 0 &\Leftrightarrow \forall \epsilon > 0, \quad \mathbb{P}(|X_n| > \epsilon) \to 0 \\ &\Leftrightarrow \mathbb{P}(X_n = 1) \to 0 \\ &\Leftrightarrow p_n \to 0. \end{aligned}$$

So any sequence $\{p_n\}$ with $p_n \to 0$ but $\sum p_k = \infty$ is such that $X_n \xrightarrow{P} 0$ but not a.s. For example $p_n = n^{-1}$.

(b) Now for any $p_n \to 0, Y_n \to 0$ a.s. This is because if $\omega > 0$ then $Y_n(\omega) = 0$ for all n such that $p_n < \omega$. Thus $Y_n(\omega) \to 0$ for all $\omega \in (0, 1]$, which is a probability 1 subset. The second Borel - Cantelli lemma fails because $\{Y_n\}$ are not independent:

$$\mathbb{P}\{Y_n = 1, Y_{n+1} = 1\} = p_{n+1} \neq p_n p_{n+1}$$

as long as $0 < p_n, p_{n+1} < 1$.

Problem 4

Apply the second B-C lemma: For any $C < \infty$, define $j(n) = \log_2([Cn \log_2 n])$. Then

$$\mathbb{P}(X_n > Cn \log_2 n) = \sum_{j: 2^j > Cn \log_2 n} 2^{-j} = 2^{-j(n)} \geq \frac{1}{Cn \log_2 n} = \infty$$

Therefore, $\mathbb{P}(X_n > Cn \log_2 n \quad i.o.) = 1$. Thus,

$$\frac{S_n}{n \log_2 n} \geq \frac{X_n}{n \log_2 n} > C \quad i.o.$$

and since C was arbitrary we can make it arbitrarily large and we have the conclusion.