

## HOMEWORK 4 SOLUTIONS

**Problem 1** First observe that

$$(1) \quad n^{-1}\mathbb{E}S_n = n^{-1} \sum_{1 \leq k \leq n} \mathbb{E}X_{n,k} = \sum_{1 \leq k \leq n} n^{-1}u\left(\frac{k}{n}\right)$$

which in particular is the Riemann sum of  $u$  for the partition  $\left\{\frac{k}{n} : 1 \leq k \leq n\right\}$ . Since  $u$  is integrable, we know that that the Riemann sum converges to the integral of  $u$ , therefore

$$(2) \quad n^{-1}\mathbb{E}S_n \xrightarrow{n \rightarrow \infty} \int_0^1 u(x) dx.$$

Define  $\xi_{n,i} = X_{n,i} - \mathbb{E}X_{n,i}$ . Now consider the fourth moment of  $\frac{S_n - \mathbb{E}S_n}{n} = n^{-1} \sum_{i=1}^n \xi_{n,i}$ .

We estimate the fourth moment:

$$(3) \quad \begin{aligned} n^{-4}\mathbb{E}\left(\sum_{1 \leq i \leq n} \xi_{n,i}\right)^4 &= n^{-4}\mathbb{E}\left(\sum_{\substack{\alpha_1 + \dots + \alpha_n = 4 \\ \alpha_i \geq 0}} \prod_{1 \leq i \leq n} \xi_{n,i}^{\alpha_i}\right) \\ &= n^{-4}\left(\sum_{\substack{\alpha_1 + \dots + \alpha_n = 4 \\ \alpha_i \geq 0}} \prod_{1 \leq i \leq n} \mathbb{E}\xi_{n,i}^{\alpha_i}\right) \end{aligned}$$

where the last equality follows from the independence (in  $i$ ) of the  $\xi_{n,i}$ . If at least one of the  $\alpha_i$  is 1, the product becomes 0, since the  $\xi_{n,i}$ 's are centered. Then the possible terms that show up in the sum (3) are:  $\xi_{n,i}^2 \xi_{n,j}^2$ .

There are exactly  $\frac{n^2 + n}{2} < Cn^2$  ways to select the indices  $i, j$  and each  $\xi_{n,i}$  is bounded, so

$$(4) \quad n^{-4}\mathbb{E}\left(\sum_{1 \leq i \leq n} \xi_{n,i}\right)^4 \leq \frac{C}{n^2}.$$

Now pick  $\epsilon$  positive. We have

$$\begin{aligned} \sum_n \mathbb{P}\left\{\left|\frac{S_n - \mathbb{E}S_n}{n}\right| > \epsilon\right\} &= \sum_n \mathbb{P}\left\{\left|\frac{S_n - \mathbb{E}S_n}{n}\right|^4 > \epsilon^4\right\} \\ &\leq \epsilon^{-4} \sum_n Cn^{-2} \text{ by Chebyshev's inequality} \\ &< +\infty. \end{aligned}$$

By the first Borel- Cantelli lemma we have that  $\frac{S_n - \mathbb{E}S_n}{n} \rightarrow 0$  a.s. To finish the proof, observe that  $\frac{S_n}{n} = \frac{S_n - \mathbb{E}S_n}{n} + \frac{\mathbb{E}S_n}{n} \rightarrow \int_0^1 u(x) dx$ .

**Problem 2:** We need a lemma first.

LEMMA 0.1. *Suppose  $N_t > 0$  for all  $t \in [0, T_n]$ . Then for all  $1 \leq m \leq n$ ,*

$$Y_0 + \dots + Y_{m-1} \geq T_m.$$

*Proof.* The proof goes by induction on  $n$ . For  $n = 1$  observe that  $Y_0 \geq T_1$  or otherwise the park is empty during  $[Y_0, T_1]$ .

Assume the lemma is true up to  $n - 1$ . Since the park is not empty during  $(T_{n-1}, T_n]$ , we must have that

$$T_n \leq \max_{0 \leq m \leq n-1} \{T_m + Y_m\}.$$

This statement just says that the  $n$ -th visitor arrived when there was at least a person inside. (Why?)

By the induction hypothesis, we get that

$$(5) \quad T_n \leq \max_{0 \leq m \leq n-1} \{Y_0 + \dots + Y_m\} = \sum_{0 \leq k \leq n-1} Y_k.$$

The SLLN implies that for a.e.  $\omega$  we have  $n^{-1}T_n(\omega) \rightarrow \mathbb{E}X_1$  and  $n^{-1} \sum_{0 \leq k \leq n-1} Y_k(\omega) \rightarrow \mathbb{E}Y_1$ . By the assumption on the expectations there exists  $N = N(\omega)$  large enough so that

$$T_n(\omega) > \sum_{0 \leq k \leq n-1} Y_k(\omega), \quad \forall n > N.$$

This contradicts the lemma (hence our original assumption on  $N_t$ .) □