HOMEWORK 2 SOLUTIONS

Problem 1
Fix $i_1, i_2, \ldots, i_k \in I$. We shall show that $Y_{i_1}, \ldots, Y_{i_k}$ are independent by showing that

$$\mathbb{P}\{Y_{i_1} \leq t_1, \ldots, Y_{i_k} \leq t_k\} = \prod_{l=1}^{k} \mathbb{P}(Y_{i_l} \leq t_l), \quad \forall t_1, t_2, \ldots, t_k \in \mathbb{R}.$$

**Step 1:** For any bounded continuous functions $f_1, f_2, \ldots, f_k$, we have

$$\mathbb{E}\left(\prod_{l=1}^{k} f_l(Y_{i_l})\right) = \prod_{l=1}^{k} \mathbb{E}f_l(Y_{i_l}).$$

$X_{n, i_l} \xrightarrow{a.s.} Y_{i_l}$ implies that $f_l(X_{n, i_l}) \xrightarrow{a.s.} f_l Y_{i_l}$, and since all the functions are bounded, Bounded Convergence Theorem (or DCT) can be applied:

$$\mathbb{E}\left(\prod_{l=1}^{k} f_l(Y_{i_l})\right) = \lim_{n \to +\infty} \mathbb{E}\left(\prod_{l=1}^{k} f_l(X_{n, i_l})\right) = \lim_{n \to +\infty} \left(\prod_{l=1}^{k} \mathbb{E}f_l(X_{n, i_l})\right) = \prod_{l=1}^{k} \mathbb{E}f_l(Y_{i_l}),$$

where the equality before last comes from the independence of $\{X_{n, i_l}\}_{l=1}^{k}$.

**Step 2:** Consider the function

$$f_{l,m} = \begin{cases} 
1, & \text{if } -\infty \leq x \leq t_l \\
-m(x - t_l) + 1, & \text{if } t_l \leq x \leq t_l + m^{-1} \\
0, & \text{if } t_l + m^{-1} \leq x < +\infty
\end{cases}$$

Then $f_{l,m}(x) \xrightarrow{m \to \infty} 1_{(-\infty, t_l]}(x)$. All functions are bounded by 1, so again by DCT:

$$\mathbb{P}\{Y_{i_1} \leq t_1, \ldots, Y_{i_k} \leq t_k\} = \lim_{m \to +\infty} \mathbb{E}\left(\prod_{l=1}^{k} f_{m,l}(Y_{i_l})\right) = \lim_{m \to +\infty} \left(\prod_{l=1}^{k} \mathbb{E}f_{m,l}(Y_{i_l})\right) = \prod_{l=1}^{k} \mathbb{P}(Y_{i_l} \leq t_l).$$

Now apply the independence criterion form Durrett.

Note that you cannot interchange limits freely in this problem. Let $1_{(0, n^{-1})} = X_n \xrightarrow{a.s.} Y = 0$ and

$$f_k = \begin{cases} 
1, & \text{if } -\infty \leq x \leq 0 \\
-kx + 1, & \text{if } 0 \leq x \leq k^{-1} \\
0, & \text{if } k^{-1} \leq x < +\infty
\end{cases}$$
Then
\[
\lim\lim_{k\to\infty} f_k(X_n) = \lim_{k\to\infty} f_k(Y) = \begin{cases} 1 & \text{if } \neq 0 \\ \lim_{n\to\infty} f_k(X_n) \end{cases}
\]

**Problem 2**

Let \( k \in \mathbb{N}; B, D \in A. \)

\[
P(T = k, X_T \in B, X_{T+1} \in D) = P(T = k, X_k \in B, X_{k+1} \in D)
= P(X_1 \notin A, ..., X_k \notin A, X_k \in A \cap B, X_{k+1} \in D)
= (1 - P(X_1 \in A))^{k-1} P(X_1 \in A) \frac{P(X_1 \in A \cap B)}{P(X_1 \in A)} P(X_1 \in D)
\]

where the last equality follows by the i.i.d. property.

Taking \( B = D = S \) gives \( P(T = k) = (1 - P(X_1 \in A))^{k-1} P(X_1 \in A). \)

Sum over \( k \in \mathbb{N} \) and take \( D = S \) to see \( P(X_T \in B) = P(X_1 \in B|X_1 \in A). \)

Finally, sum over \( k \in \mathbb{N} \) and take \( B = S \) to see \( P(X_{T+1} \in D) = P(X_1 \in D). \)

These were the marginal distributions of \( T, X_T \) and \( X_{T+1}. \) The identity above shows the independence.

**Problem 3**

The Borel \( \sigma- \) algebra, \( \mathcal{B}_{(0,1)} \) is generated (for example) by open intervals, since every open set is a countable union of open intervals. Let \((a, b) \subseteq (0, 1). \) If \( a \geq n^{-1} \) for some \( n \in \mathbb{N}, \) then \((a, b) \in \mathcal{A}_n. \) If \( a = 0, \) then take \( n \) large enough so that \( n^{-1} < b. \) Then, \((0, b) = (0, n^{-1}) \cup [n^{-1}, b] \in \mathcal{A}_n. \) Thus, each open interval lies in \( \bigcup_n \mathcal{A}_n. \)

Suppose that such a probability measure \( \mu \) exists. Then

\[
0 = \mu(\emptyset) = \mu \left( \bigcap_n \left(0, n^{-1}\right) \right) = \lim_{n \to +\infty} \mu(0, n^{-1}) = \lim_{n \to +\infty} \mu_n(0, n^{-1}) = 1,
\]
which gives the desired contradiction.