HOMEWORK 2 SOLUTIONS

Problem 1

Fix $i_1, i_2, ..., i_k \in \mathcal{I}$. We shall show that $Y_{i_1}, ...Y_{i_k}$ are independent by showing that

$$\mathbb{P}\{Y_{i_1} \le t_i, ..., Y_{i_k} \le t_k\} = \prod_{l=1}^{k} \mathbb{P}(Y_{i_l} \le t_l), \quad \forall t_1, t_2, ..t_k \in \mathbb{R}.$$

Step1: For any bounded continuous functions $f_1, f_2, ..., f_k$, we have

$$\mathbb{E}\left(\prod_{l=1}^{k} f_l(Y_{i_l})\right) = \prod_{l=1}^{k} \mathbb{E} f_l(Y_{i_l}).$$

$X_{n,i_l} \overset{a.s.}{\to} Y_{i_l}$ implies that $f_l(X_{n,i_l}) \overset{a.s.}{\to} f_l Y_{i_l}$, and since all the functions are bounded, Bounded Convergence Theorem (or DCT) can be applied:

$$\mathbb{E}\left(\prod_{l=1}^{k} f_l(Y_{i_l})\right) = \lim_{n\to+\infty} \mathbb{E}\left(\prod_{l=1}^{k} f_l(X_{n,i_l})\right) = \lim_{n\to+\infty}\left(\prod_{l=1}^{k} \mathbb{E} f_l(X_{n,i_l})\right) = \prod_{l=1}^{k} \mathbb{E} f_l(Y_{i_l}), \text{where}$$

the equality before last comes from the independence of $\{X_{n,i_l}\}_{l=1}^{k}$.

Step 2: Consider the function

$$f_{l,m} = \begin{cases} 1, & \text{if } -\infty \le x \le t_l \\ -m(x-t_l)+1, & \text{if } t_l \le x \le t_l + m^{-1} \\ 0, & \text{if } t_l + m^{-1} \le x < +\infty \end{cases}$$

Then $f_{l,m}(x) \overset{m\to\infty}{\longrightarrow} 1_{(-\infty,t_l]}(x)$. All functions are bounded by 1, so again by DCT:

$$\mathbb{P}\{Y_{i_1} \le t_i, ..., Y_{i_k} \le t_k\} = \lim_{m\to\infty} \mathbb{E}\left(\prod_{l=1}^{k} f_{m,l}(Y_{i_l})\right) = \lim_{m\to\infty}\left(\prod_{l=1}^{k} \mathbb{E} f_{m,l}(Y_{i_l})\right)$$
$$= \prod_{l=1}^{k} \mathbb{P}(Y_{i_l} \le t_l).$$

Now apply the independence criterion form Durrett.

Note that you cannot interchange limits freely in this problem. Let $1_{(0,n^{-1})} = X_n \overset{a.s.}{\to} Y = 0$ and

$$f_k = \begin{cases} 1, & \text{if } -\infty \le x \le 0 \\ -kx+1, & \text{if } 0 \le x \le k^{-1} \\ 0, & \text{if } k^{-1} \le x < +\infty \end{cases}$$

Then

$$\begin{aligned}\lim_k \lim_n f_k(X_n) &= \lim_k f_k(Y)\\ &= 1\\ &\neq 0\\ &= \lim_n \lim_k f_k(X_n)\end{aligned}$$

Problem 2

Let $k \in \mathbb{N}; B, D \in \mathcal{A}$.

$$\begin{aligned}\mathbb{P}\{T = k, X_T \in B, X_{T+1} \in D\} &= \mathbb{P}\{T = k, X_k \in B, X_{k+1} \in D\}\\ &= \mathbb{P}\{X_1 \notin A, ..., X_{k-1} \notin A, X_k \in A \cap B, X_{k+1} \in D\}\\ &= (1 - \mathbb{P}(X_1 \in A))^{k-1}\, \mathbb{P}(X_1 \in A)\frac{\mathbb{P}(X_1 \in A \cap B)}{\mathbb{P}(X_1 \in A)}\mathbb{P}(X_1 \in D)\end{aligned}$$

where the last equality follows by the i.i.d. property.

Taking $B = D = S$ gives $\mathbb{P}(T = k) = (1 - \mathbb{P}(X_1 \in A))^{k-1}\mathbb{P}(X_1 \in A)$.

Sum over $k \in \mathbb{N}$ and take $D = S$ to see $\mathbb{P}(X_T \in B) = \mathbb{P}(X_1 \in B | X_1 \in A)$.

Finally, sum over $k \in \mathbb{N}$ and take $B = S$ to see $\mathbb{P}(X_{T+1} \in D) = \mathbb{P}(X_1 \in D)$.

These were the marginal distributions of T, X_T and X_{T+1}. The identity above shows the independence.

Problem 3

The Borel $\sigma-$ algebra, $\mathcal{B}_{(0,1)}$ is generated (for example) by open intervals, since every open set is a countable union of open intervals. Let $(a, b) \subseteq (0, 1)$. If $a \geq n^{-1}$ for some $n \in \mathbb{N}$, then $(a, b) \in \mathcal{A}_n$. If $a = 0$, then take n large enough so that $n^{-1} < b$. Then, $(0, b) = (0, n^{-1}) \cup [n^{-1}, b] \in \mathcal{A}_n$. Thus, each open interval lies in $\bigcup_n \mathcal{A}_n$.

Suppose that such a probability measure μ exists. Then $0 = \mu(\varnothing) = \mu\left(\bigcap_n (0, n^{-1})\right) = \lim_{n\to+\infty} \mu(0, n^{-1}) = \lim_{n\to+\infty} \mu_n(0, n^{-1}) = 1$, which gives the desired contradiction.