831 Theory of Probability Fall 2009 Homework 2

Due Tuesday, September 22

1. Let $\{X_{n,i} : n \in \mathbb{N}, i \in \mathcal{I}\}$ be real-valued random variables on $(\Omega, \mathcal{F}, P)$. Assume that for each $n \in \mathbb{N}$, the variables $\{X_{n,i} : i \in \mathcal{I}\}$ are independent. Assume that for each $i \in \mathcal{I}$ there is a real-valued random variable Y_i on $(\Omega, \mathcal{F}, P)$ such that $X_{n,i} \to Y_i$ a.s. as $n \to \infty$. Show that $\{Y_i : i \in \mathcal{I}\}$ are independent. (Hint: you may find it convenient to consider random variables of the type $f(X_{n,i})$ where f is a bounded, continuous function.)

2. Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. random variables with values in some measurable space $(S, \mathcal{A})$. Let $A \in \mathcal{A}$ be a set such that $P(X_1 \in A) > 0$. Let

$$T = \inf\{n \geq 1 : X_n \in A\}$$

be the index of the first sample that lies in the set A. Find the distributions of T, X_T and X_{T+1}. What can you say about the independence of (T, X_T, X_{T+1})?

3. Inspired by the Kolmogorov Extension Theorem, we might think that whenever we specify consistent probability measures μ_n on increasing σ-algebras $\mathcal{A}_n$, then we get a probability measure on the generated σ-algebra $\mathcal{A} = \sigma(\bigcup_n \mathcal{A}_n)$. But consider this example.

Let the space be the open unit interval $(0, 1)$. For each $n \in \mathbb{N}$, let $\mathcal{A}_n$ be the smallest σ-algebra that contains the set $(0, \frac{1}{n})$ and the Borel subsets of $[\frac{1}{n}, 1)$. On the measurable space $((0, 1), \mathcal{A}_n)$ define the probability measure μ_n by $\mu_n\{(0, \frac{1}{n})\} = 1$, $\mu_n\{[\frac{1}{n}, 1)\} = 0$. Show that the σ-algebras $\mathcal{A}_n$ generate the Borel σ-algebra $\mathcal{B}_{(0,1)}$, but there is no probability measure μ on $((0, 1), \mathcal{B}_{(0,1)})$ such that, for all n, the restriction of μ to $\mathcal{A}_n$ is μ_n.