

## 831 Theory of Probability Fall 2009 Homework 3

**Due Thursday, October 1**

1. Let  $\theta > 0$ . Show that, for  $a > 0$ ,

$$\lim_{n \rightarrow \infty} \left\{ e^{-n\theta} \sum_{k: 0 \leq k \leq na} \frac{(n\theta)^k}{k!} \right\} = \begin{cases} 0, & \text{if } a < \theta \\ 1, & \text{if } a > \theta. \end{cases}$$

Hint: apply the WLLN to Poisson random variables.

2. For each  $n \in \mathbb{N}$  let  $\{X_{n,k} : 1 \leq k \leq n\}$  be IID random variables such that  $0 \leq X_{n,k} \leq C$  (same constant  $C$  for all  $n$  and  $k$ ). Let  $S_n = X_{n,1} + X_{n,2} + \cdots + X_{n,n}$ . Assume that

$$ES_n \rightarrow \infty \quad \text{and} \quad \text{Var } S_n \rightarrow \infty.$$

Show that  $S_n \rightarrow \infty$  in probability. The natural way to interpret this statement is that for any  $k < \infty$ ,  $P\{S_n \leq k\} \rightarrow 0$  as  $n \rightarrow \infty$ . Hint: since  $ES_n \rightarrow \infty$ , try to control the distance of  $S_n$  to its mean with Chebychev's inequality.

3. Let  $\{p_n\}$  be given numbers such that  $0 \leq p_n \leq 1$ . Let  $\{X_n\}$  be independent random variables with distributions

$$P(X_n = 1) = 1 - P(X_n = 0) = p_n.$$

(a) Find the condition on  $\{p_n\}$  that is equivalent to  $X_n \rightarrow 0$  almost surely. Find the condition on  $\{p_n\}$  that is equivalent to  $X_n \rightarrow 0$  in probability. Give an example where  $X_n \rightarrow 0$  in probability but not almost surely.

(b) On the sample space  $\Omega = [0, 1]$  with  $P =$  Lebesgue measure, let  $Y_n(\omega) = \mathbf{1}_{[0, p_n]}(\omega)$ . Show that, even if  $\{p_n\}$  satisfies  $\sum P(Y_n = 1) = \infty$ , the conclusion of the second Borel-Cantelli Lemma can fail. Why is that?

4. Exercise 6.17 from p. 54 of Durrett. That is, show that in the St. Petersburg game,  $\overline{\lim}_{n \rightarrow \infty} S_n / (n \log_2 n) = \infty$  a.s., in contrast with the weak law we discovered.