831 Theory of Probability Fall 2009 Homework 4

Due Tuesday, October 13

1. Monte Carlo integration. Assume given a continuous function u on $[0,1]$ such that $0 \le u(x) \le 1$. We use u to create an array of $\{0,1\}$-valued random variables as follows: let $\{X_{n,k} : 1 \le k \le n,\ n \in \mathbb{N}\}$ be independent random variables with marginal distributions

$$P(X_{n,k} = 0) = 1 - u(k/n) \quad \text{and} \quad P(X_{n,k} = 1) = u(k/n).$$

Let $S_n = X_{n,1} + X_{n,2} + \cdots + X_{n,n}$ be the sum of row n of the array. Show that $n^{-1}S_n$ converges to some constant almost surely and identify the limit.

Hint: Look at the convergence of $n^{-1}ES_n + n^{-1}(S_n - ES_n)$. Estimate a high enough moment of $n^{-1}(S_n - ES_n)$.

2. Let $\{X_k\}_{k\ge1}$ be nonnegative i.i.d. random variables, and $T_0 = 0$, $T_n = X_1 + \cdots + X_n$. Let $\{Y_j\}_{j\ge0}$ be another sequence of nonnegative i.i.d. random variables. The $\{X_k\}$ and $\{Y_j\}$ are defined on the same probability space, but their joint distribution is immaterial for this exercise. Let $D_n = T_n + Y_n$ for $n \ge 0$, and

$$N_t = \sum_{n\ge0} \mathbf{1}\{T_n \le t < D_n\} \qquad \text{for } t \in [0,\infty).$$

Assuming that $EY_0 < EX_1 < \infty$, show that

$$P\{\ N_t = 0 \text{ for some } t > 0\ \} = 1.$$

Here is a story that gives some meaning to this mathematical set-up. A park opens up at time $t = 0$ and the first visitor (visitor number 0) arrives also at time 0. After that, visitor n arrives to the park at time T_n. Y_n is the amount of time that visitor n lingers at the park, and D_n is the moment visitor n leaves the park. N_t is the number of visitors at the park at time t. The question is to show that eventually there comes a point in time when the park is empty, assuming that the mean interarrival time is larger than the mean sojourn time.

This exercise may lead you to some kind of longwinded, technically complex attempt. But it is not that hard. Draw a picture (not of the park!) and think a little. An excessively complicated solution may be impossible to grade.