831 Theory of Probability Fall 2009 Homework 5

Due Tuesday, October 27

1. Let $\{X_n\}$, X be real random variables.

(a) Suppose $X_n \to X$ in probability. Show that then also $X_n \xrightarrow{d} X$.

(b) Suppose X_n converges in distribution to a constant c. Show that then $X_n \to c$ also in probability.

2. Let $f_n(x) = 1 - \cos(2\pi n x)$ for $n \in \mathbb{N}$ and $x \in [0,1]$. Verify that f_n is the density of a probability measure μ_n on $[0,1]$. Is there a weak limit $\mu_n \xrightarrow{d} \mu$? Either show convergence and identify μ or prove that the sequence does not converge. (Sketch f_n for large n if you are baffled.)

3. Let $S_n = X_1 + \cdots + X_n$ be a sum of i.i.d. mean zero random variables with a finite variance. Show that $\overline{\lim}_{n\to\infty} S_n/\sqrt{n} = \infty$ a.s. (Hint: CLT and Kolmogorov's 0-1 law.)

4. Recall the coupon collector's problem: for a fixed n, $Y_1, Y_2, Y_3, \ldots$ are i.i.d. uniform on $\{1, 2, \ldots, n\}$, and

$$\tau_k^n = \inf\{m : |\{Y_1, Y_2, \ldots, Y_m\}| = k\}$$

is the time when we first see k different coupons. Fix $0 < \theta < 1$, and let $S_n = \tau_{[n\theta]}^n$, the time by which we have seen fraction θ of the coupons. First find the asymptotics of the mean and variance of S_n: that is, find constants c and σ^2 and exponents α and β such that

$$E(S_n) \sim n^\alpha c \quad \text{and} \quad \mathrm{Var}(S_n) \sim n^\beta \sigma^2 \quad \text{as } n \to \infty.$$

Then prove a central limit theorem: find exponent γ such that $n^{-\gamma}(S_n - ES_n)$ converges weakly to a nondegenerate normal distribution. (It is essential here that $\theta < 1$. $a_n \sim b_n$ means $a_n/b_n \to 1$.)