831 Theory of Probability Fall 2009 In-class Exam

Please hand in exactly 3 problems. You may appeal to theorems covered in class and basic facts from analysis.

1. Let $\{X_n\}_{n\geq 1}$ be independent, a.s. finite random variables. Show that

$$\sup_n X_n < \infty \text{ a.s.} \quad \text{if and only if} \quad \exists c < \infty \text{ such that } \sum_n P(X_n > c) < \infty.$$

2.(a) Let $\{X_{n,j} : 1 \leq j \leq n < \infty\}$ be Bernoulli random variables with distributions

$$P(X_{n,j} = 1) = p_n = 1 - P(X_{n,j} = 0).$$

Assume there exists a constant $\delta > 0$ such that $\delta \leq p_n \leq 1 - \delta$ for all n. Find constants $a_n \in \mathbb{R}$, $b_n > 0$ and a nondegenerate limit distribution μ such that $b_n^{-1}(S_n - a_n)$ converges weakly to the distribution μ.

(b) Let $Y_1, Y_2, Y_3, \ldots$ be i.i.d. uniform on $\{1, 2, \ldots, n\}$. Let

$$\sigma_n = \inf\{k \geq 2 : Y_k = Y_m \text{ for some } 1 \leq m < k\}$$

be the first time we get a repeated sample. Find an exponent γ and a nondegenerate distribution μ such that, as $n \to \infty$, $n^{-\gamma}\sigma_n$ converges weakly to the distribution μ. (Hint: look at tail probabilities $P(n^{-\gamma}\sigma_n \geq x)$.)

3. Let X, Y, Z be random variables on some probability space. Recall that $P(X \in A \,|\, Y, Z) = P(X \in A \,|\, \sigma\{Y, Z\})$ is the conditional probability of the event $\{X \in A\}$, given the σ-algebra $\sigma\{Y, Z\}$ generated by Y and Z.

We say that X *and* Y *are independent, given* Z if for all Borel sets A, $B \subseteq \mathbb{R}$,

$$P(X \in A, Y \in B \,|\, Z) = P(X \in A \,|\, Z) \cdot P(Y \in B \,|\, Z) \qquad \text{a.s.}$$

Show that this condition is equivalent to having

$$P(X \in A \,|\, Y, Z) = P(X \in A \,|\, Z) \qquad \text{a.s.}$$

for all Borel sets A.

4. Let $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ be simple symmetric random walk on $\mathbb{Z}$, in other words $\{X_i\}$ are i.i.d. random variables with distribution

$$P(X_i = 1) = P(X_i = -1) = 1/2.$$

Fix integers $a < 0 < b$, and let T be the first time the walk hits a or b:

$$T = \inf\{n \geq 1 : \; S_n = a \text{ or } S_n = b \}.$$

You can take for granted that $T < \infty$ a.s. since we proved this in class.

(a) Show that $S_n^2 - n$ is a martingale.

(b) Calculate $E(T)$. (The answer is a simple formula in terms of a and b. Be careful and precise about applying any theorems you may want to use.)