Please hand in exactly 3 problems. You may appeal to theorems covered in class and basic facts from analysis.

1. Let \( \{X_n\}_{n \geq 1} \) be independent, a.s. finite random variables. Show that 
\[
\sup_n X_n < \infty \text{ a.s. if and only if } \exists c < \infty \text{ such that } \sum_n P(X_n > c) < \infty.
\]

2. (a) Let \( \{X_{n,j} : 1 \leq j \leq n < \infty\} \) be Bernoulli random variables with distributions
\[P(X_{n,j} = 1) = p_n = 1 - P(X_{n,j} = 0).\]
Assume there exists a constant \( \delta > 0 \) such that \( \delta \leq p_n \leq 1 - \delta \) for all \( n \). Find constants \( a_n \in \mathbb{R} \), \( b_n > 0 \) and a nondegenerate limit distribution \( \mu \) such that \( b_n^{-1}(S_n - a_n) \) converges weakly to the distribution \( \mu \).

(b) Let \( Y_1, Y_2, Y_3, \ldots \) be i.i.d. uniform on \( \{1, 2, \ldots, n\} \). Let
\[\sigma_n = \inf\{k \geq 2 : Y_k = Y_m \text{ for some } 1 \leq m < k\}\]
be the first time we get a repeated sample. Find an exponent \( \gamma \) and a nondegenerate distribution \( \mu \) such that, as \( n \to \infty \), \( n^{-\gamma} \sigma_n \) converges weakly to the distribution \( \mu \). (Hint: look at tail probabilities \( P(n^{-\gamma} \sigma_n \geq x) \).)

3. Let \( X, Y, Z \) be random variables on some probability space. Recall that \( P(X \in A \mid Y, Z) = P(X \in A \mid \sigma\{Y, Z\}) \) is the conditional probability of the event \( \{X \in A\} \), given the \( \sigma \)-algebra \( \sigma\{Y, Z\} \) generated by \( Y \) and \( Z \).

We say that \( X \) and \( Y \) are independent, given \( Z \) if for all Borel sets \( A, B \subseteq \mathbb{R} \),
\[P(X \in A, Y \in B \mid Z) = P(X \in A \mid Z) \cdot P(Y \in B \mid Z) \quad \text{a.s.}\]
Show that this condition is equivalent to having
\[P(X \in A \mid Y, Z) = P(X \in A \mid Z) \quad \text{a.s.}\]
for all Borel sets \( A \).
4. Let $S_0 = 0, S_n = X_1 + \cdots + X_n$ be simple symmetric random walk on $\mathbb{Z}$, in other words $\{X_i\}$ are i.i.d. random variables with distribution

$$P(X_i = 1) = P(X_i = -1) = 1/2.$$ 

Fix integers $a < 0 < b$, and let $T$ be the first time the walk hits $a$ or $b$:

$$T = \inf\{n \geq 1 : S_n = a \text{ or } S_n = b \}.$$

You can take for granted that $T < \infty$ a.s. since we proved this in class.

(a) Show that $S_n^2 - n$ is a martingale.

(b) Calculate $E(T)$. (The answer is a simple formula in terms of $a$ and $b$. Be careful and precise about applying any theorems you may want to use.)