1. Suppose that for each \( j \in \mathbb{N} \) the process \((X^j_n)_{n \in \mathbb{Z}^+}\) is stationary. Assume that for each \( n \) we have the limits 
\[
(X^j_0, X^j_1, \ldots, X^j_n) \longrightarrow (X_0, X_1, \ldots, X_n) \quad \text{as} \quad j \rightarrow \infty.
\]
Is \((X_n)_{n \in \mathbb{Z}^+}\) stationary? Consider here convergence almost surely, in probability and in distribution.

2. This exercise practices a simple coupling and answers the following question. Suppose that for each \( j \in \mathbb{N} \) the process \((X^j_n)_{n \in \mathbb{Z}^+}\) is stationary and ergodic. Suppose that for each \( n \in \mathbb{Z}^+ \), \( X^j_n \rightarrow X_n \) a.s. as \( j \rightarrow \infty \). Is the limit process \((X_n)_{n \in \mathbb{Z}^+}\) necessarily ergodic?

(a) Let \( \{U_n\}_{n \in \mathbb{N}} \) be i.i.d. Uniform(0, 1)-random variables. Construct \( \{0, 1\}\)-valued processes \((Y^j_n)_{n \in \mathbb{Z}^+}\) with the following recipe. Let \( Y^j_0 \) take values 0 and 1 with equal probability. For each \( j \in \mathbb{N} \) set \( Y^j_0 = Y_0 \), and then for \( n \geq 1 \),
\[
Y^j_n = \begin{cases} 
Y^j_{n-1} & \text{if } U_n \leq 1 - 2^{-j} \\
1 - Y^j_{n-1} & \text{if } U_n > 1 - 2^{-j}.
\end{cases}
\]
Show that for each \( j \in \mathbb{N} \), \((Y^j_n)_{n \in \mathbb{Z}^+}\) is a stationary Markov chain and identify its transition matrix and invariant distribution.

Define a process \((Y^j_n)_{n \in \mathbb{Z}^+}\) such that \( Y^j_n \rightarrow Y_n \) a.s. as \( j \rightarrow \infty \). In fact, show that with probability 1, for every \( n \) we have the equality \((Y^j_0, Y^j_1, \ldots, Y^j_n) = (Y_0, Y_1, \ldots, Y_n)\) for all large enough \( j \).

(b) Answer the question posed above about whether a.s. limits preserve ergodicity.

3. **Exchangeability and ergodicity.** For this exercise, assume that \( P \) is a probability measure on the sequence space \( \Omega = \mathbb{R}^\mathbb{N} \) with product \( \sigma \)-algebra \( \mathcal{F} \), and \( P \) is exchangeable. By this we mean that \( P(D) = P(\pi^{-1}D) \) for all events \( D \in \mathcal{F} \) and all finite permutations \( \pi \). Equivalently, the coordinate process \( X_n(\omega) = \omega_n \) is exchangeable. As before, \( \mathcal{E} \) is the exchangeable
σ-field, in other words the σ-field of events $A \in \mathcal{F}$ such that $\pi^{-1}A = A$ for all finite permutations $\pi$. In the fall we proved that $P$ is an i.i.d. product measure iff $\mathcal{E}$ is trivial under $P$. [“$\mathcal{E}$ is trivial” means that all $A \in \mathcal{E}$ satisfy $P(A) = 0$ or 1.]

(a) Show that $P$ is shift-invariant.

(b) Show that if $P$ is ergodic under the shift, then $\mathcal{E}$ is trivial under $P$. Conclude that i.i.d. processes are the only exchangeable processes that are ergodic.

Hint: Approximate $A \in \mathcal{E}$ by a finite-dimensional set $B$. You may use Exercise 3.1 from Durrett’s Appendix without proving it, as it is really part of basic measure theory.

(c) Let $\omega^* = (1, 0, 1, 0, \ldots)$, its shift $\theta \omega^* = (0, 1, 0, 1, \ldots)$, and define a probability measure $Q$ on $\Omega$ by $Q = (\delta_{\omega^*} + \delta_{\theta \omega^*})/2$. Show that $Q$ is ergodic but $\mathcal{E}$ is not trivial under $Q$. So part (b) above is not true in general, only for exchangeable measures.