Fluctuation bounds for a class of asymmetric zero range processes

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1. The general question
2. The totally asymmetric zero range process in one dimension
3. The main result and its broader context
4. Second class particles and other ingredients of the proof

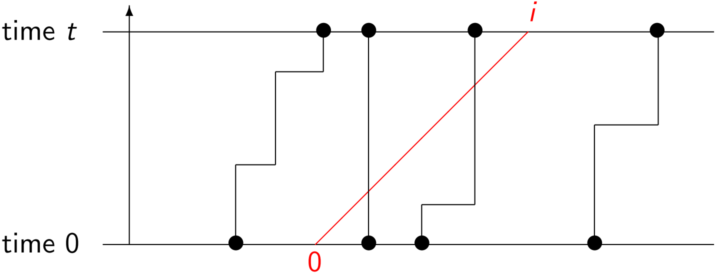
The question addressed

Imagine particles moving randomly on the integer lattice $\mathbb{Z}$.

Net left-to-right current:

$$\begin{aligned} J_i(t) \;=\; & \#\{\text{particles that start in } (-\infty,0] \text{ at } t=0 \text{ and} \\ & \qquad \text{reside in } (i,\infty) \text{ at time } t\} \\ & - \#\{\text{particles that start in } (0,\infty) \text{ at } t=0 \text{ and} \\ & \qquad \text{reside in } (-\infty,i] \text{ at time } t\} \end{aligned}$$

Illustration: space-time particle trajectories and current



$$J_i(t) = -2$$

The question addressed

$J_{vt}(t)$ = current seen by observer moving at fixed speed v.

Question: In a stationary process, what can we say about the fluctuations of $J_{vt}(t)$ as $t \to \infty$?

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Our result (joint with M. Balázs and J. Komjáthy, Budapest) is the order of magnitude of

$$\mathbf{Var}[J_{vt}(t)] \qquad \text{as } t \to \infty$$

in a class of totally asymmetric zero range processes (TAZRP).

Description of the totally asymmetric ZRP

Zero range process is a continuous time Markov process that represents the motion of particles on $\mathbb{Z}$.

No a priori limit on number of particles per site.

State. $\eta_i(t) =$ number of particles at site i at time t.

Particle configuration $\eta(t) = (\eta_i(t))_{i\in\mathbb{Z}}$. State space $\mathbb{Z}_+^{\mathbb{Z}}$.

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Dynamics. Given nondecreasing function $g : \mathbb{Z}_+ \to \mathbb{R}_+$, $g(0) = 0$, $g(k) > 0$ for $k > 0$.

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Assume g nondecreasing.

Description of the totally asymmetric ZRP

Examples.

1. $g(k) = \mathbf{1}\{k > 0\}$. Constant rate ZRP same as M/M/1 queues in series.
2. $g(k) = pk$. Independent walks with jump rate p to the right.

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Generator. For bounded cylinder functions φ on $\mathbb{Z}_+^{\mathbb{Z}}$

$$L\varphi(\eta) = \sum_{i\in\mathbb{Z}} g(\eta_i)[\varphi(\eta^{i,i+1}) - \varphi(\eta)]$$

where $\eta^{i,i+1} = \eta - \delta_i + \delta_{i+1}$.

ZRP invariant distributions

Family of invariant distributions $\{\nu^\rho : 0 \leq \rho < \infty\}$ on $\mathbb{Z}_+^{\mathbb{Z}}$ indexed by density $\rho = E^\rho(\eta_i)$.

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Definition. On $\mathbb{Z}_+$ define $\lambda^\theta(k) = \frac{1}{Z_\theta} \frac{e^{\theta k}}{g(k)!}$ for θ s.t.

$$Z_\theta = \sum_k \frac{e^{\theta k}}{g(k)!} < \infty. \quad \text{Density } \rho(\theta) = \sum_k k\lambda^\theta(k).$$

Reparametrize in terms of density: $\nu_0^\rho = \lambda^{\theta(\rho)}$.

Product measure on state space: $\nu^\rho(d\eta) = \bigotimes_{i\in\mathbb{Z}} \nu_0^\rho(d\eta_i)$.

Flux and hydrodynamic limit

$$\begin{aligned} \text{The } \text{flux } H(\rho) &= E^{\rho}[\text{rate of particle flow across a fixed edge}] \\ &= E^{\rho}[g(\eta_i)]. \end{aligned}$$

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Hydrodynamic limit informally

Under suitable hypotheses on initial conditions,

$$n^{-1} \sum_{i=na}^{nb} \eta_i^n(nt) \longrightarrow \int_a^b u(t,x)\,dx \qquad \text{as} \quad n \to \infty$$

where u solves the conservation law $u_t + H(u)_x = 0$.

Assumption on jump rate

Recall: jump rate g nondecreasing, $g(k) > 0$ for $k > 0$.

Assumption ($\star$)

$\exists\ 0 < r < 1$ such that $g(k+1) - g(k) \leq r(g(k) - g(k-1))$.

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Example. $g(k) = 1 - \exp(-ak^b)$ with $a > 0$, $b \geq 1$.

$g(k)$ can also be constant from some k_0 onwards.

Any concave, not linear jump rate g guarantees $H''(\rho) < 0\ \forall\ \rho$.

Characteristic speed

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Significance for the PDE: if u is a smooth solution of

$$u_t + H(u)_x = 0$$

and $\dot{x} = H'(u(t,x))$, then $u(t,x(t))$ is constant in time.

Result: variance bound for current across characteristic

$(\star)\ \exists\ 0 < r < 1$ s.t. $g(k+1) - g(k) \le r(g(k) - g(k-1))$

$V^\rho = H'(\rho)$

Stationary process: $\eta(t) \sim \nu^\rho$ at each time t

Theorem (Balázs, Komjáthy, S.)

Assume $(\star)$ and consider stationary TAZRP at density $0 < \rho < \infty$. Then $\exists$ constants $0 < t(\rho), C(\rho) < \infty$ such that

$$t \ge t(\rho) \implies \frac{1}{C(\rho)} t^{2/3} \le \mathbf{Var}^\rho\big[J_{V^\rho t}(t)\big] \le C(\rho) t^{2/3}$$

Corollary for other speeds $v \neq V^\rho$

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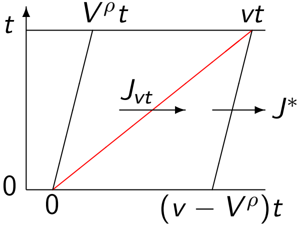

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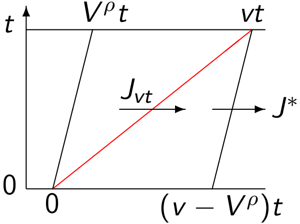

$$J^* = J_{vt} + \sum_{i=1}^{(v-V^\rho)t} \eta_i(0)$$

after centering:

$$\bar{J}_{vt} = \sum_{i=1}^{(v-V^\rho)t} \bar{\eta}_i(0) - \bar{J}^*$$

$$\sim t^{1/2} \cdot \mathcal{N}(0, \sigma^2) + O(t^{1/3})$$

Context: quest for universality

Expected: for asymmetric particle systems (particles have drift) curvature of H at ρ determines type of fluctuations of current across characteristic.

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$H''(\rho) \neq 0$ $t^{1/3}$ scaling, limits related to Tracy-Widom laws;
weak limit: TASEP (Johansson, Ferrari-Spohn), ASEP (Tracy-Widom)
moment bounds: ASEP, this class of TAZRP, BLP with exponential jump rate (Balázs, Komjáthy, S.)

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H linear	$t^{1/4}$ scaling, limits Gaussian related to fBM(1/4); examples: independent walks, random average process (Balázs, Joseph, Kumar, Rassoul-Agha, S.)

Second class particle

Return to discuss the result:

Theorem (Balázs, Komjáthy, S.)

Assume $(\star)$ and consider stationary TAZRP at density $0 < \rho < \infty$. Then for large t,

$$C^{-1}\, t^{2/3} \leq \mathbf{Var}^{\rho}\big[J_{V^{\rho} t}(t)\big] \leq C\, t^{2/3}$$

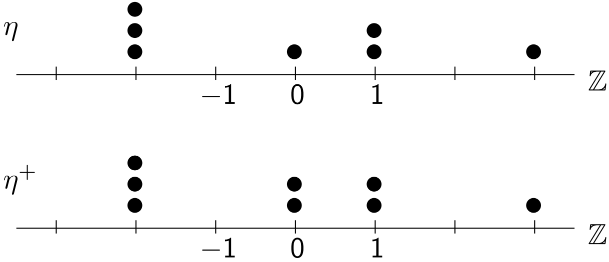
The proof turns on the connection of current with discrepancies or second class particles.

A single second class particle

Basic coupling of two ZRP's (η, η^+) with one discrepancy.

Move together from site i with rate $g(\eta_i)$,

move only η^+ with rate $g(\eta_i^+) - g(\eta_i)$.



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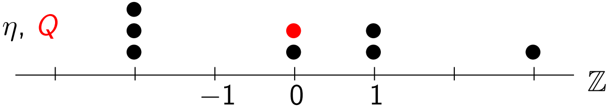
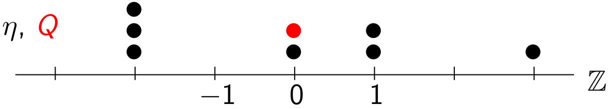


A single second class particle

Basic coupling of two ZRP's (η, η^+) with one discrepancy.
Equivalently: $\eta(t)$ a usual ZRP, second class particle $Q(t)$ jumps with rate $g(\eta_Q + 1) - g(\eta_Q)$.



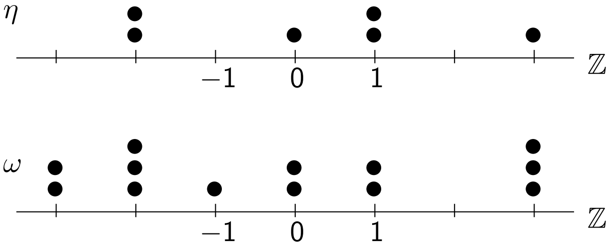
$$\eta^+(t) = \eta(t) + \delta_{Q(t)}$$

Arbitrarily many second class particles

Basic coupling of two ZRP's (η, ω) with $\eta_i \leq \omega_i$.

Move together from site i with rate $g(\eta_i)$,

move only ω with rate $g(\omega_i) - g(\eta_i)$.



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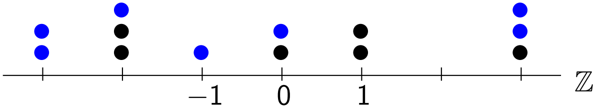
Basic coupling of two ZRP's (η, ω) with $\eta_i \leq \omega_i$.

Equivalently: $\eta(t)$ usual ZRP, an $\omega - \eta$ second class particle is moved at rate $g(\omega_i) - g(\eta_i)$.



$$\eta = \{\text{black particles}\}$$

$$\omega = \eta + \{\text{blue particles}\}$$

Current, covariances and second class particles

Two key identities:

$$\mathbf{Var}^{\rho}[J_z(t)] = \sum_x |z - x| \mathbf{Cov}^{\rho}[\eta_x(t), \eta_0(0)]$$

$$\mathbf{Cov}^{\rho}[\eta_x(t), \eta_0(0)] = \mathbf{Var}^{\rho}(\eta_0)\, \widehat{\mathbf{P}}^{\rho}\{Q(t) = x\}$$

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In the coupling $\widehat{\mathbf{P}}^\rho$ of $\eta^+ = \eta + \delta_Q$, the new initial distribution $\hat{\nu}^\rho$ of η is invariant ν^ρ perturbed at 0:

$$\hat{\nu}^\rho(d\eta) = \Big(\bigotimes_{i\neq 0} \nu_0^\rho(d\eta_i)\Big) \otimes \hat{\nu}_0^\rho(d\eta_0)$$

where

$$\hat{\nu}_0^\rho(k) = \frac{1}{\mathbf{Var}^\rho(\eta_0)} \sum_{m=k+1}^{\infty} (m-\rho)\nu_0^\rho(m), \qquad k \in \mathbb{Z}_+$$

Current and second class particles

Combine to give $\mathbf{Var}^{\rho}[J_z(t)] = C(\rho)\,\widehat{\mathbf{E}}^{\rho}|Q(t) - z|$

So task is to get moment bound on second class particle. Initially $Q(0) = 0$.

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Theorem (Balázs, Komjáthy, S.)

Assume $(\star)$ and consider stationary TAZRP at density $0 < \rho < \infty$.

Then $\exists$ constants $0 < t(\rho), C(\rho) < \infty$ such that for $1 \le m < 3$ and $t \ge t(\rho)$

$$\frac{1}{C(\rho)} t^{2m/3} \;\le\; \widehat{\mathbf{E}}^\rho\big[\,|Q(t) - V^\rho t|^m\,\big] \;\le\; \frac{C(\rho)}{3 - m}\, t^{2m/3}$$

Second class particle

It is also the case that $\widehat{\mathbf{E}}^{\rho} Q(t) = V^{\rho} t$ and so theorem implies

$$\widehat{\mathbf{Var}}^{\rho}\left[Q(t)\right] \sim t^{4/3}$$

i.e. Q is superdiffusive.

Another corollary is a weak LLN: $t^{-1} Q(t) \xrightarrow{P} V^{\rho}$.

The task is now to prove the previous theorem, for it implies the bounds on the variance of the current.

Streamlined view of the proofs

These steps used for both upper and lower bound:

1. deviation in Q $\implies$ deviation in certain currents
2. deviation in current bounded by variance of current (by Chebyshev)
3. variance of current $=$ moment of Q, and the "inequality is closed" (recall $\mathbf{Var}[J_{V^\rho t}(t)] = C\,\widehat{\mathbf{E}}|Q(t) - V^\rho t|$)

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For the upper bound these steps result in

$$\widehat{\mathbf{P}}\{\,|Q(t) - V^\rho t| > u\,\} \;\le\; \frac{Ct^2}{u^4}\,\widehat{\mathbf{E}}|Q(t) - V^\rho t| \;+\; \frac{Ct^2}{u^3}$$

which suffices for the conclusion.

Coupling to control a 2nd class particle

Let us discuss the step

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Coupling to control a 2nd class particle

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1. deviation in $Q \implies$ deviation in certain currents

To achieve this we embed Q in a positive density of second class particles with the help of a label process.

- an auxiliary density $\lambda < \rho$
- a coupling of two processes ω at density ρ and η at density λ
- ordered labeling of $\omega - \eta$ second class particles:

  $\cdots \leq X_{-1}(t) \leq X_0(t) \leq X_1(t) \leq \cdots$ with $X_0(0) = 0$
- construct a label process $y(t) \in \mathbb{Z}$ such that $X_{y(t)}(t)$ behaves like a second class particle in the process $\omega^- = \omega - \delta_{X_y}$. Then we can represent $Q(t)$ with $X_{y(t)}(t)$.

Label process

Evolution of label $y(t)$ superimposed on background process $(\eta, \{X_m\})$. After a jump that puts X_y at site i, y is updated randomly, within the labels at i, according to

$$y := \begin{cases} a & \text{with prob. } \dfrac{g(\omega_i - 1) - g(\eta_i)}{g(\omega_i) - g(\eta_i)} \\[2ex] b & \text{with prob. } \dfrac{g(\omega_i) - g(\omega_i - 1)}{g(\omega_i) - g(\eta_i)} \end{cases}$$

where $a =$ minimal label on site i, $b =$ maximal label on site i.

This guarantees (ω^-, X_y) has the rates of (ω^-, Q)

Tail bound on label process

To be of use label process needs bounds! Assumption

$(\star)\ \exists\ 0 < r < 1$ s.t. $g(k+1) - g(k) \le r(g(k) - g(k-1))$

is crucially used for this lemma:

Lemma

Under $(\star)$, $\mathbf{P}\{y(t) \ge k\} \le r^k$ for all $k \in \mathbb{Z}_+$ and $t \ge 0$.

Remark. The other parts of the proof work for all TAZRP's with concave jump rate. Obtaining this bound is the tricky step.

Sketch of upper bound

▸ centering

Let $u > 0$, $k = bu^2/t$, $\lambda = \rho - cu/t$. UB proof start off with:

$$\begin{aligned}\widehat{\mathbf{P}}^{\rho}\{Q(t) > V^{\rho}t + u\} &\le \mathbf{P}\{y(t) \ge k\} \;+\; \mathbf{P}\{X_k(t) > V^{\rho}t + u\} \\ &\le r^k \;+\; \mathbf{P}\big\{J^{\omega}_{V^{\rho}t+u}(t) - J^{\eta}_{V^{\rho}t+u}(t) > -k\big\} \\ &\le r^k \;+\; \frac{Ct^2}{u^4}\Big(\mathbf{Var}^{\rho}[J_{V^{\rho}t+u}(t)] + \mathbf{Var}^{\lambda}[J_{V^{\rho}t+u}(t)]\Big) \\ &\le r^k \;+\; \frac{Ct^2}{u^4}\Big(\widehat{\mathbf{E}}^{\rho}|Q(t) - V^{\rho}t| + \widehat{\mathbf{E}}^{\lambda}|Q(t) - V^{\rho}t| + u\Big)\end{aligned}$$

Label construction developed further to allow comparison of second class particles in the two densities $\lambda < \rho$.

Sketch of upper bound, cont'd

This leads to the inequality

$$\widehat{\mathbf{P}}^{\rho}\{\, Q(t) > V^{\rho}t + u \,\} \;\le\; \frac{Ct^2}{u^4}\,\widehat{\mathbf{E}}^{\rho}|Q(t) - V^{\rho}t| \;+\; \frac{Ct^2}{u^3}$$

Same bound works for the left tail.

In the end this leads to the upper bound

$$\widehat{\mathbf{E}}^{\rho}\big[\,|Q(t) - V^{\rho}t|^{m}\,\big] \;\le\; \frac{C(\rho)}{3-m}\,t^{2m/3}$$

The lower bound proof uses similar ideas but is more technical.

Summary

A class of zero range processes has been verified to obey the expected “universal” behavior of asymmetric systems with convex/concave flux, as far as the order of magnitude of fluctuations is concerned.

The novelty is that the result has now been established beyond the class of exclusion-type processes that benefit from special rigid properties of a combinatorial flavor.

The hypotheses that this proof uses are in the form of estimates rather than structural properties, so there is hope that the class of processes can be further enlarged.

Centering of currents

Recall: $V^\rho = H'(\rho)$, $\rho - \lambda = cu/t$.

$$\begin{aligned}
&\mathbf{E}^\rho[J_{V^\rho t+u}(t)] \;-\; \mathbf{E}^\lambda[J_{V^\rho t+u}(t)] \\
&= \bigl\{tH(\rho) - \rho(tH'(\rho) + u)\bigr\} \;-\; \bigl\{tH(\lambda) - \lambda(tH'(\rho) + u)\bigr\} \\
&\approx -t\frac{H''(\rho)}{2}(\rho - \lambda)^2 \;-\; u(\rho - \lambda) \\
&= \Bigl(-\frac{1}{2}c^2 H''(\rho) - c\Bigr)\frac{u^2}{t} \;=\; -c_1\frac{u^2}{t}
\end{aligned}$$

for constant $c_1 > 0$ by choosing $c > 0$ small.

▸ back to UB sketch