Large deviations and fluctuation exponents for some polymer models

Timo Seppäläinen

Department of Mathematics
University of Wisconsin-Madison

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2 Large deviations

3 Fluctuation exponents
   - KPZ equation
   - Log-gamma polymer
Directed polymer in a random environment

simple random walk path \((x(t), t), t \in \mathbb{Z}_+\)
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\(\mathbb{P}\) probability distribution on \(\omega\), often \(\{\omega(x, t)\}\) i.i.d.
Key quantities again:

- Quenched measure $Q_n \{ x(\cdot) \} = Z_n^{-1} \exp \left\{ \beta \sum_{t=1}^n \omega(x(t), t) \right\}$

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- Dependence on $\beta$ and $d$
Large deviations

**Question:** describe quenched limit \( \lim_{n \to \infty} n^{-1} \log Z_n \) (\( \mathbb{P} \)-a.s.)
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Generalize: \( E_0 = \) expectation under background RW \( X_n \) on \( \mathbb{Z}^\nu \).
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Introduced shift \( (T X \omega)_y = \omega_{x+y}, \text{ steps } Z_k = X_k - X_{k-1} \in \mathcal{R}, \)

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\( g(\omega, z_{1,\ell}) \) is a function on \( \Omega_\ell = \Omega \times \mathcal{R}^\ell \).
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Defines kernel $p$ on $\Omega_{\ell}$: $p(\eta, S_z \eta) = |\mathcal{R}|^{-1}$. 
Entropy

For $\mu \in \mathcal{M}_1(\Omega_\ell)$, $q$ Markov kernel on $\Omega_\ell$, usual relative entropy on $\Omega_\ell^2$:

$$H(\mu \times q \mid \mu \times p) = \int_{\Omega_\ell} \sum_{z \in \mathcal{R}} q(\eta, S_z \eta) \log \frac{q(\eta, S_z \eta)}{p(\eta, S_z \eta)} \mu(d\eta).$$
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$$H_\mathbb{P}(\mu) = \begin{cases} \inf \left\{ H(\mu \times q \mid \mu \times p) : \mu q = \mu \right\} & \text{if } \mu_0 \ll \mathbb{P} \\ \infty & \text{otherwise.} \end{cases}$$

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$H_\mathbb{P}$ is convex but not lower semicontinuous.
Assumptions.

- Environment \( \{ \omega_x \} \) IID under \( \mathbb{P} \).
- \( g \) local function on \( \Omega_\ell \), \( \mathbb{E}|g|^p < \infty \) for some \( p > \nu \).
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**Theorem.** (Rassoul-Agha, S, Yilmaz) Deterministic limit

$$\Lambda(g) = \lim_{n \to \infty} n^{-1} \log E_0[e^{nR_n(g)}]$$
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**Remarks.**

- With higher moments of \( g \) admit mixing \( \mathbb{P} \).
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**Remarks.**

- With higher moments of $g$ admit mixing $\mathbb{P}$.
- $\Lambda(g) > -\infty$.
- IID directed $+$ above moment $\Rightarrow \Lambda(g)$ finite.
Quenched weak LDP (large deviation principle) under $Q_n$.

\[ Q_n(A) = \frac{1}{E_0[e^{nR_n(g)}]} \ E_0[e^{nR_n(g)} 1_A(\omega, Z_1, \infty)] \]
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Rate function $I(\mu) = \inf_{c>0} \{ H_p(\mu) - E^\mu(g \wedge c) + \Lambda(g) \}$. 
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**Theorem.** (RSY) Assumptions as above and $\Lambda(g)$ finite. Then $\mathbb{P}$-a.s. for compact $F \subseteq \mathcal{M}_1(\Omega_\ell)$ and open $G \subseteq \mathcal{M}_1(\Omega_\ell)$:

$$\lim_{n \to \infty} n^{-1} \log Q_n\{R_n \in F\} \leq - \inf_{\mu \in F} I(\mu)$$

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IID environment, directed walk: full LDP holds.
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**Early results:** diffusive behavior for \( d \geq 3 \) and small \( \beta > 0 \):

- 1988 Imbrie and Spencer: \( n^{-1}E^Q(|x(n)|^2) \to c \) \( \mathbb{P}\text{-a.s.} \)
- 1989 Bolthausen: quenched CLT for \( n^{-1/2}x(n) \).
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**In the opposite direction**: if $d = 1, 2$, or $d \geq 3$ and $\beta$ large enough, then $\exists \ c > 0$ s.t.

$$\lim_{n \to \infty} \max_z Q_n\{x(n) = z\} \geq c \quad \mathbb{P}$-a.s.$$

(Carmona and Hu 2002, Comets, Shiga, and Yoshida 2003)
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**Results:** these exact exponents for three particular 1+1 dimensional models.
Earlier results for $d = 1$ exponents

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- **Gaussian RW in Gaussian potential**: Petermann 2000 $\zeta \geq 3/5$, Mejane 2004 $\zeta \leq 3/4$
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3. **Continuum directed polymer,** or Hopf-Cole solution of the Kardar-Parisi-Zhang (KPZ) equation:
   - (i) Initial height function given by two-sided Brownian motion (joint with M. Balázs and J. Quastel).
   - (ii) Narrow wedge initial condition (Amir, Corwin, Quastel).
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1. Log-gamma polymer: $\beta = 1$ and $e^{-\omega(x,t)} \sim \text{Gamma}$, plus appropriate boundary conditions.

2. Polymer in a Brownian environment (joint with B. Valkó) Model introduced by O’Connell and Yor 2001.

3. Continuum directed polymer, or Hopf-Cole solution of the Kardar-Parisi-Zhang (KPZ) equation:
   
   (i) Initial height function given by two-sided Brownian motion (joint with M. Balázs and J. Quastel).
   
   (ii) Narrow wedge initial condition (Amir, Corwin, Quastel).

Next details on (3.i), then details on (1).
Hopf-Cole solution to KPZ equation

KPZ eqn for height function $h(t, x)$ of a 1+1 dim interface:

$$
 h_t = \frac{1}{2} h_{xx} - \frac{1}{2} (h_x)^2 + \dot{W}
$$

where $\dot{W} = \text{Gaussian space-time white noise.}$
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Initial height \( h(0, x) = \) two-sided Brownian motion for \( x \in \mathbb{R} \).
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Initial height $h(0, x) =$ two-sided Brownian motion for $x \in \mathbb{R}$.

$Z = \exp(-h)$ satisfies $Z_t = \frac{1}{2} Z_{xx} - Z \dot{W}$ that can be solved.
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Define $h = -\log Z$, the **Hopf-Cole solution** of KPZ.
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Bertini-Giacomin (1997): $h$ can be obtained as a weak limit via a smoothing and renormalization of KPZ.
$\zeta_\varepsilon(t,x)$ height process of weakly asymmetric simple exclusion s.t.

$$\zeta_\varepsilon(x + 1) - \zeta_\varepsilon(x) = \pm 1$$
WASEP connection

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Jumps:

\[
\zeta_\varepsilon(x) \rightarrow \begin{cases} 
\zeta_\varepsilon(x) + 2 & \text{with rate } \frac{1}{2} + \sqrt{\varepsilon} \text{ if } \zeta_\varepsilon(x) \text{ is a local min} \\
\zeta_\varepsilon(x) - 2 & \text{with rate } \frac{1}{2} \text{ if } \zeta_\varepsilon(x) \text{ is a local max}
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h_\varepsilon(t, x) = \varepsilon^{1/2} \left( \zeta_\varepsilon(\varepsilon^{-2} t, [\varepsilon^{-1} x]) - \nu_\varepsilon t \right)
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\]

**Theorem** (Bertini-Giacomin 1997) As \( \varepsilon \downarrow 0 \), \( h_\varepsilon \Rightarrow h \)
From coupling arguments for WASEP

\[ C_1 t^{2/3} \leq \text{Var}(h_\varepsilon(t,0)) \leq C_2 t^{2/3} \]
Fluctuation bounds

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**Theorem** (Balázs-Quastel-S) For the Hopf-Cole solution of KPZ,

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Lower bound comes from control of rescaled correlations

\[ S_\varepsilon(t, x) = 4\varepsilon^{-1} \text{Cov}[\eta(\varepsilon^{-2} t, \varepsilon^{-1} x) , \eta(0, 0)] \]
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where \( \eta(t, x) \in \{0, 1\} \) is the occupation variable of WASEP
Rescaled correlations again:

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E\left[ \langle \varphi', h_\varepsilon(t) \rangle \langle \psi', h_\varepsilon(0) \rangle \right] = \frac{1}{2} \int \left[ \int \varphi \left( \frac{y + x}{2} \right) \psi \left( \frac{y - x}{2} \right) \, dy \right] S_\varepsilon(t, x) \, dx
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On the right \( S_\varepsilon(t, x) \, dx \Rightarrow S(t, dx) \) with control of moments:

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\int |x|^m S_\varepsilon(t, x) \, dx \sim O(t^{2m/3}), \quad 1 \leq m < 3.
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(Second class particle estimate.)
After $\varepsilon \downarrow 0$ limit

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With some control over tails we arrive at the result:

$$\text{Var}(h(t, 0)) = \int |x| \, S(t, dx) \sim O(t^{2/3}).$$
1+1 dimensional lattice polymer with log-gamma weights

Fix both endpoints.
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quenched measure \( Q_{m,n}(x.) = Z_{m,n}^{-1} \prod_{k=1}^{m+n} Y_{x_k} \)

averaged measure \( P_{m,n}(x.) = \mathbb{E} Q_{m,n}(x.) \)
Weight distributions

- Parameters \(0 < \theta < \mu\).
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- $\mathbb{E}(\log U) = -\Psi_0(\theta)$ and $\text{Var}(\log U) = \Psi_1(\theta)$
Variance bounds for log $Z$

With $0 < \theta < \mu$ fixed and $N \uparrow \infty$ assume

$$|m - N\psi_1(\mu - \theta)| \leq CN^{2/3} \quad \text{and} \quad |n - N\psi_1(\theta)| \leq CN^{2/3} \quad (1)$$
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**Theorem**

For $(m, n)$ as in (1), $C_1 N^{2/3} \leq \text{Var}(\log Z_{m,n}) \leq C_2 N^{2/3}$. 


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**Theorem**

Suppose $n = \Psi_1(\theta)N$ and $m = \Psi_1(\mu - \theta)N + \gamma N^\alpha$ with $\gamma > 0$, $\alpha > 2/3$. Then

$$ N^{-\alpha/2} \left\{ \log Z_{m,n} - \mathbb{E}(\log Z_{m,n}) \right\} \Rightarrow \mathcal{N}(0, \gamma \Psi_1(\theta)) $$
Fluctuation bounds for path

\( v_0(j) = \) leftmost, \( v_1(j) = \) rightmost point of \( x \). on horizontal line:

\[
\begin{align*}
v_0(j) &= \min \{ i \in \{0, \ldots, m\} : \exists k : x_k = (i, j) \} \\
v_1(j) &= \max \{ i \in \{0, \ldots, m\} : \exists k : x_k = (i, j) \}
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\]

**Theorem**

Assume \((m, n)\) as previously and \(0 < \tau < 1\). Then

(a) \( P\left\{ v_0(\lfloor \tau n \rfloor) < \tau m - bN^{2/3} \text{ or } v_1(\lfloor \tau n \rfloor) > \tau m + bN^{2/3} \right\} \leq \frac{C}{b^3} \)

(b) \( \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that} \)

\[
\lim_{N \to \infty} P\left\{ \exists k \text{ such that } |x_k - (\tau m, \tau n)| \leq \delta N^{2/3} \right\} \leq \varepsilon.
\]
Results for log-gamma polymer summarized

With reciprocals of gammas for weights, both endpoints of the polymer fixed and the right boundary conditions on the axes, we have identified the one-dimensional exponents

\[ \zeta = \frac{2}{3} \quad \text{and} \quad \chi = \frac{1}{3}. \]
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Next some key points of the proof.
Burke property for log-gamma polymer with boundary

Given initial weights \((i, j \in \mathbb{N})\):

\[
U_{i,0}^{-1} \sim \text{Gamma}(\theta), \quad V_{0,j}^{-1} \sim \text{Gamma}(\mu - \theta)
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Compute \(Z_{m,n}\) for all \((m, n) \in \mathbb{Z}^2_+\) and then define

\[
U_{m,n} = \frac{Z_{m,n}}{Z_{m-1,n}} \quad V_{m,n} = \frac{Z_{m,n}}{Z_{m,n-1}} \quad X_{m,n} = \left( \frac{Z_{m,n}}{Z_{m+1,n}} + \frac{Z_{m,n}}{Z_{m,n+1}} \right)^{-1}
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\]

For an undirected edge \(f\):

\[
T_f = \begin{cases} 
U_x & f = \{x - e_1, x\} \\
V_x & f = \{x - e_2, x\}
\end{cases}
\]
• down-right path \((z_k)\) with edges \(f_k = \{z_{k-1}, z_k\}, \ k \in \mathbb{Z}\)

• interior points \(\mathcal{I}\) of path \((z_k)\)
down-right path \((z_k)\) with edges \(f_k = \{z_{k-1}, z_k\}, k \in \mathbb{Z}\)

- interior points \(I\) of path \((z_k)\)

**Theorem**

Variables \(\{T_{f_k}, X_z : k \in \mathbb{Z}, z \in I\}\) are independent with marginals

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“Burke property” because the analogous property for last-passage is a generalization of Burke’s Theorem for M/M/1 queues, via the last-passage representation of M/M/1 queues in series.
Proof of Burke property

Induction on $I$ by flipping a growth corner:

\[
U' = Y(1 + U/V) \quad V' = Y(1 + V/U) \\
X = (U^{-1} + V^{-1})^{-1}
\]
Proof of Burke property

Induction on $I$ by flipping a growth corner:

Lemma. Given that $(U, V, Y)$ are independent positive r.v.'s, $(U', V', X) \overset{d}= (U, V, Y)$ iff $(U, V, Y)$ have the gamma distr's.

Proof. “if” part by computation, “only if” part from a characterization of gamma due to Lukacs (1955). □
Proof of Burke property

Induction on \( I \) by flipping a growth corner:

\[
\begin{align*}
U' &= Y(1 + U/V) & V' &= Y(1 + V/U) \\
X &= (U^{-1} + V^{-1})^{-1}
\end{align*}
\]

**Lemma.** Given that \((U, V, Y)\) are independent positive r.v.’s, \((U', V', X) \overset{d}{=} (U, V, Y)\) iff \((U, V, Y)\) have the gamma distr’s.

**Proof.** “if” part by computation, “only if” part from a characterization of gamma due to Lukacs (1955). \(\square\)

This gives all \((z_k)\) with finite \( I \). General case follows.
Proof of off-characteristic CLT

Recall that

\[
\begin{align*}
    n &= \psi_1(\theta)N \\
    m &= \psi_1(\mu - \theta)N + \gamma N^\alpha
\end{align*}
\]

\(\gamma > 0, \; \alpha > 2/3.\)
Proof of off-characteristic CLT

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Set \(m_1 = \lfloor \Psi_1(\mu - \theta) N \rfloor.\)
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Proof of off-characteristic CLT

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    n &= \Psi_1(\theta)N \\
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\[\gamma > 0, \ \alpha > 2/3.\]

Set \(m_1 = \lfloor \Psi_1(\mu - \theta)N \rfloor\). Since \(Z_{m,n} = Z_{m_1,n} \cdot \prod_{i=m_1+1}^m U_{i,n}\)

\[
N^{-\alpha/2} \log Z_{m,n} = N^{-\alpha/2} \log Z_{m_1,n} + N^{-\alpha/2} \sum_{i=m_1+1}^m \log U_{i,n}
\]

First term on the right is \(O(N^{1/3-\alpha/2}) \to 0\). Second term is a sum of order \(N^\alpha\) i.i.d. terms. \(\square\)
Variance identity

Exit point of path from $x$-axis

$$\xi_x = \max\{k \geq 0 : x_i = (i, 0) \text{ for } 0 \leq i \leq k\}$$
Variance identity

Exit point of path from \( x \)-axis

\[ \xi_x = \max\{ k \geq 0 : x_i = (i, 0) \text{ for } 0 \leq i \leq k \} \]

For \( \theta, x > 0 \) define positive function

\[ L(\theta, x) = \int_0^x (\psi_0(\theta) - \log y) x^{-\theta} y^{\theta-1} e^{x-y} \, dy \]
Variance identity

Exit point of path from x-axis

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For $\theta, x > 0$ define positive function

$$L(\theta, x) = \int_0^x (\Psi_0(\theta) - \log y) x^{-\theta} y^{\theta-1} e^{x-y} dy$$

**Theorem.** For the model with boundary,

$$\text{Var}[\log Z_{m,n}] = n\Psi_1(\mu - \theta) - m\Psi_1(\theta) + 2 E_{m,n} \left[ \sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1}) \right]$$
Variance identity, sketch of proof

\[ N = \log Z_{m,n} - \log Z_{0,n} \]

\[ W = \log Z_{0,n} \]

\[ S = \log Z_{m,0} \]

\[ E = \log Z_{m,n} - \log Z_{m,0} \]
Variance identity, sketch of proof

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\[ W = \log Z_{0,n} \]
\[ S = \log Z_{m,0} \]
\[ E = \log Z_{m,n} - \log Z_{m,0} \]

\[ \var[\log Z_{m,n}] = \var(W + N) \]
\[ = \var(W) + \var(N) + 2 \cov(W, N) \]
\[ = \var(W) + \var(N) + 2 \cov(S + E - N, N) \]
\[ = \var(W) - \var(N) + 2 \cov(S, N) \quad (E, N \ind) \]
\[ = n\psi_1(\mu - \theta) - m\psi_1(\theta) + 2 \cov(S, N). \]
To differentiate w.r.t. parameter $\theta$ of $S$ while keeping $W$ fixed, introduce a separate parameter $\rho (= \mu - \theta)$ for $W$.

$$-\text{Cov}(S, N) = \frac{\partial}{\partial \theta} \mathbb{E}(N)$$
To differentiate w.r.t. parameter $\theta$ of $S$ while keeping $W$ fixed, introduce a separate parameter $\rho \ (= \mu - \theta)$ for $W$.

$$-\text{Cov}(S, N) = \frac{\partial}{\partial \theta} \mathbb{E}(N) = \tilde{\mathbb{E}} \left[ \frac{\partial}{\partial \theta} \log Z_{m,n}(\theta) \right]$$
To differentiate w.r.t. parameter $\theta$ of $S$ while keeping $W$ fixed, introduce a separate parameter $\rho (= \mu - \theta)$ for $W$.

$$-\text{Cov}(S, N) = \frac{\partial}{\partial \theta} E(N) = \tilde{E}\left[ \frac{\partial}{\partial \theta} \log Z_{m,n}(\theta) \right]$$

when $Z_{m,n}(\theta) = \sum_{x \in \Pi_{m,n}} \prod_{i=1}^{\xi_x} H_{\theta}(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k}$ with

$$\eta_i \sim \text{IID Unif}(0, 1), \quad H_{\theta}(\eta) = F_{\theta}^{-1}(\eta), \quad F_{\theta}(x) = \int_0^x \frac{y^{\theta-1}e^{-y}}{\Gamma(\theta)} \, dy.$$
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$$\eta_i \sim \text{IID Unif}(0, 1), \quad H_{\theta}(\eta) = F_{\theta}^{-1}(\eta), \quad F_{\theta}(x) = \int_0^x y^{\theta-1} e^{-y} \frac{1}{\Gamma(\theta)} \, dy.$$
Together:

$$\text{Var}[\log Z_{m,n}] = n\Psi_1(\mu - \theta) - m\Psi_1(\theta) + 2\text{Cov}(S, N)$$

$$= n\Psi_1(\mu - \theta) - m\Psi_1(\theta) + 2E_{m,n}\left[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\right].$$

This was the claimed formula. $\square$
The argument develops an inequality that controls both $\log Z$ and $\xi_x$ simultaneously. Introduce an auxiliary parameter $\lambda = \theta - bu/N$. 
Sketch of upper bound proof

The argument develops an inequality that controls both \( \log Z \) and \( \xi_x \) simultaneously. Introduce an auxiliary parameter \( \lambda = \theta - bu/N \). The weight of a path \( x \) such that \( \xi_x > 0 \) satisfies

\[
W(\theta) = \prod_{i=1}^{\xi_x} H_\theta(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k}
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$$W(\theta) = \prod_{i=1}^{\xi_x} H_\theta(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k} = W(\lambda) \cdot \prod_{i=1}^{\xi_x} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)}.$$
Sketch of upper bound proof

The argument develops an inequality that controls both log $Z$ and $\xi_x$ simultaneously. Introduce an auxiliary parameter $\lambda = \theta - bu/N$. The weight of a path $x$, such that $\xi_x > 0$ satisfies

$$W(\theta) = \prod_{i=1}^{m+n} H_\theta(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{\xi_x} Y_{x_k} = W(\lambda) \cdot \prod_{i=1}^{\xi_x} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)}.$$ 

Since $H_\lambda(\eta) \leq H_\theta(\eta)$,

$$Q^{\theta,\omega}\{\xi_x \geq u\} = \frac{1}{Z(\theta)} \sum_x 1\{\xi_x \geq u\} W(\theta) \leq \frac{Z(\lambda)}{Z(\theta)} \cdot \prod_{i=1}^{\lfloor u \rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)}.$$
For $1 \leq u \leq \delta N$ and $0 < s < \delta$,

$$\mathbb{P}\left[ Q^\omega \{ \xi_x \geq u \} \geq e^{-su^2/N} \right] \leq \mathbb{P}\left\{ \prod_{i=1}^{\lfloor u \rfloor} \frac{H_{\lambda}(\eta_i)}{H_{\theta}(\eta_i)} \geq \alpha \right\}$$

$$+ \mathbb{P}\left( \frac{Z(\lambda)}{Z(\theta)} \geq \alpha^{-1} e^{-su^2/N} \right).$$
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Choose $\alpha$ right. Bound these probabilities with Chebychev which brings $\text{Var}(\log Z)$ into play. In the characteristic rectangle $\text{Var}(\log Z)$ can be bounded by $E(\xi_x)$. The end result is this inequality:

$$\mathbb{P}[Q^\omega \{ \xi_x \geq u \} \geq e^{-su^2/N}] \leq \frac{CN^2}{u^4} E(\xi_x) + \frac{CN^2}{u^3}$$
For $1 \leq u \leq \delta N$ and $0 < s < \delta$, 

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P\left[ Q^\omega \{ \xi_x \geq u \} \geq e^{-su^2/N} \right] \leq P \left\{ \prod_{i=1}^{[u]} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)} \geq \alpha \right\} 
+ P \left( \frac{Z(\lambda)}{Z(\theta)} \geq \alpha^{-1} e^{-su^2/N} \right).$$

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Handle $u \geq \delta N$ with large deviation estimates. In the end, integration gives the moment bounds.
For $1 \leq u \leq \delta N$ and $0 < s < \delta$,

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\mathbb{P}\left[ Q^\omega \{ \xi_x \geq u \} \geq e^{-su^2/N} \right] \leq \mathbb{P}\left\{ \prod_{i=1}^{\lfloor u \rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)} \geq \alpha \right\} + \mathbb{P}\left( \frac{Z(\lambda)}{Z(\theta)} \geq \alpha^{-1} e^{-su^2/N} \right).
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\]

Handle $u \geq \delta N$ with large deviation estimates. In the end, integration gives the moment bounds. **END.**
Polymer in a Brownian environment

Environment: independent Brownian motions $B_1, B_2, \ldots, B_n$

Partition function (without boundary conditions):

$$Z_{n,t}(\beta) = \int_{0 < s_1 < \cdots < s_{n-1} < t} \exp\left[ \beta \left( B_1(s_1) + B_2(s_2) - B_2(s_1) + B_3(s_3) - B_3(s_2) + \cdots + B_n(t) - B_n(s_{n-1}) \right) \right] ds_1, s_{n-1}$$