Scaling exponents for certain 1+1 dimensional directed polymers

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1 Introduction

2 KPZ equation

3 Log-gamma polymer
Directed polymer in a random environment

simple random walk path \((x(t), t), t \in \mathbb{Z}_+\)

space \(\mathbb{Z}^d\)

time \(\mathbb{N}\)
Directed polymer in a random environment

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Inverse temperature \(\beta > 0\)

Quenched probability measure on paths

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Q_n\{x(\cdot)\} = \frac{1}{Z_n} \exp \left\{ \beta \sum_{t=1}^{n} \omega(x(t), t) \right\}
\]
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Z_n = \sum_{x(\cdot)} \exp\left\{\beta \sum_{t=1}^{n} \omega(x(t), t)\right\}
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(summed over all \(n\)-paths)
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\(\mathbb{P}\) probability distribution on \(\omega\), often \(\{\omega(x, t)\}\) i.i.d.
**General question:** Behavior of model as $\beta > 0$ and dimension $d$ vary. Especially whether $x(\cdot)$ is diffusive or not, that is, does it scale like standard RW.
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**Early results:** diffusive behavior for $d \geq 3$ and small $\beta > 0$:

1988 Imbrie and Spencer: $n^{-1}E^Q(|x(n)|^2) \to c$  $\mathbb{P}$-a.s.
1989 Bolthausen: quenched CLT for $n^{-1/2}x(n)$. 
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In the opposite direction: if $d = 1, 2$, or $d \geq 3$ and $\beta$ large enough, then $\exists c > 0$ s.t.

$$\lim_{n \to \infty} \max_z Q_n\{x(n) = z\} \geq c \quad \mathbb{P}$-a.s.$$

(Comets, Shiga, Yoshida 2003)
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- Conjecture for $d = 1$: $\zeta = 2/3$ and $\chi = 1/3$. 
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Our results: these exact exponents for certain 1+1 dimensional models with particular weight distributions and boundary conditions.
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Models for which we can show $\zeta = 2/3$ and $\chi = 1/3$

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(3) Continuum directed polymer, or Hopf-Cole solution of the Kardar-Parisi-Zhang (KPZ) equation with initial height function given by two-sided Brownian motion. (Joint with M. Balázs and J. Quastel.)
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Next details on (3), then more details on (1).
Polymer in a Brownian environment

**Environment**: independent Brownian motions $B_1, B_2, \ldots, B_n$

**Partition function (without boundary conditions)**:

\[
Z_{n,t}(\beta) = \int_{0<s_1<\cdots<s_{n-1}<t} \exp \left[ \beta \left( B_1(s_1) + B_2(s_2) - B_2(s_1) + B_3(s_3) - B_3(s_2) + \cdots + B_n(t) - B_n(s_{n-1}) \right) \right] ds_{1,n-1}
\]
Hopf-Cole solution to KPZ equation

KPZ eqn for height function $h(t,x)$ of a 1+1 dim interface:

$$h_t = \frac{1}{2} h_{xx} - \frac{1}{2} (h_x)^2 + \dot{W}$$

where $\dot{W} = $ Gaussian space-time white noise.
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Bertini-Giacomin (1997): \( h \) can be obtained as a weak limit via a smoothing and renormalization of KPZ.
WASEP connection

\( \zeta_\varepsilon(t, x) \) height process of weakly asymmetric simple exclusion s.t.

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\zeta_\varepsilon(x + 1) - \zeta_\varepsilon(x) = \pm 1
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Jumps:

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\zeta_\varepsilon(x) \longrightarrow \begin{cases} 
\zeta_\varepsilon(x) + 2 & \text{with rate } 1/2 + \sqrt{\varepsilon} \text{ if } \zeta_\varepsilon(x) \text{ is a local min} \\
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h_\varepsilon(t, x) = \varepsilon^{1/2} \left( \zeta_\varepsilon(\varepsilon^{-2} t, [\varepsilon^{-1} x]) - v_\varepsilon t \right)
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\[ h_\varepsilon(t, x) = \varepsilon^{1/2} (\zeta_\varepsilon(\varepsilon^{-2} t, [\varepsilon^{-1} x]) - v_\varepsilon t) \]

**Thm.** As \( \varepsilon \downarrow 0 \), \( h_\varepsilon \Rightarrow h \) (Bertini-Giacomin 1997).
Fluctuation bounds

From coupling arguments for WASEP

\[ C_1 t^{2/3} \leq \text{Var}(h_\varepsilon(t, 0)) \leq C_2 t^{2/3} \]
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**Thm.** (Balázs-Quastel-S) For the Hopf-Cole solution of KPZ,

$$C_1 t^{2/3} \leq \text{Var}(h(t,0)) \leq C_2 t^{2/3}$$
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The lower bound comes from control of rescaled correlations

\[ S_\varepsilon(t,x) = \varepsilon^{-1} \text{Cov}[\eta(\varepsilon^{-2} t, \varepsilon^{-1} x), \eta(0,0)] \]
Rescaled correlations:

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\[ S_\varepsilon(t, x) = \varepsilon^{-1} \text{Cov}[\eta(\varepsilon^{-2}t, \varepsilon^{-1}x), \eta(0, 0)] \]

\[ S_\varepsilon(t, x) dx \Rightarrow S(t, dx) \text{ with control of moments:} \]

\[ \int |x|^m S_\varepsilon(t, x) \, dx \sim O(t^{2m/3}), \quad 1 \leq m < 3. \]

(A second class particle estimate.)
Rescaled correlations:

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\[ S(t, dx) = \frac{1}{2} \partial_{xx} \text{Var}(h(t, x)) \text{ as distributions.} \]
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With some control over tails we arrive at

\[ \text{Var}(h(t, 0)) = \int |x| \, S(t, dx) \sim O(t^{2/3}). \]
Polymer in first quadrant with fixed endpoints

We turn the picture 45 degrees. Polymer is an up-right path from $(0,0)$ to $(m,n)$ in $\mathbb{Z}^2_+$. 
**Notation**

- $\Pi_{m,n} = \{ \text{up-right paths } (x_k) \text{ from } (0,0) \text{ to } (m,n) \}$
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- \( \text{Fix } \beta = 1. \ Y_{i,j} = e^{\omega(i,j)} \text{ independent.} \)
- \( \text{Environment } (Y_{i,j} : (i,j) \in \mathbb{Z}_+^2) \text{ with distribution } \mathbb{P}. \)
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- \( Z_{m,n} = \sum_{x \in \Pi_{m,n}} \prod_{k=1}^{m+n} Y_{x_k} \quad Q_{m,n}(x.) = \frac{1}{Z_{m,n}} \prod_{k=1}^{m+n} Y_{x_k} \)
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- Fix $\beta = 1$. $Y_{i,j} = e^{\omega(i,j)}$ independent.
- Environment $\left( Y_{i,j} : (i,j) \in \mathbb{Z}^2_+ \right)$ with distribution $\mathbb{P}$.
- $Z_{m,n} = \sum_{x_\cdot \in \Pi_{m,n}} \prod_{k=1}^{m+n} Y_{x_k}$ $Q_{m,n}(x_\cdot) = \frac{1}{Z_{m,n}} \prod_{k=1}^{m+n} Y_{x_k}$
- $P_{m,n}(x_\cdot) = \mathbb{E} Q_{m,n}(x_\cdot)$ (averaged measure)
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- Environment $(Y_{i,j} : (i,j) \in \mathbb{Z}_+^2)$ with distribution $\mathbb{P}$.
- $Z_{m,n} = \sum_{x_* \in \Pi_{m,n}} \prod_{k=1}^{m+n} Y_{x_k}$
- $Q_{m,n}(x_*) = \frac{1}{Z_{m,n}} \prod_{k=1}^{m+n} Y_{x_k}$
- $P_{m,n}(x_*) = \mathbb{E}Q_{m,n}(x_*)$ (averaged measure)
- Boundary weights also $U_{i,0} = Y_{i,0}$ and $V_{0,j} = Y_{0,j}$. 
Weight distributions

For \( i, j \in \mathbb{N} \):

- \( U_{i,0}^{-1} \sim \text{Gamma}(\theta) \)
- \( V_{0,j}^{-1} \sim \text{Gamma}(\mu - \theta) \)
- \( Y_{i,j}^{-1} \sim \text{Gamma}(\mu) \)

- parameters \( 0 < \theta < \mu \)
- Gamma(\theta) density: \( \Gamma(\theta)^{-1}x^{\theta-1}e^{-x} \) on \( \mathbb{R}_+ \)
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- parameters \( 0 < \theta < \mu \)
- \( \text{Gamma}(\theta) \) density: \( \Gamma(\theta)^{-1} x^{\theta - 1} e^{-x} \) on \( \mathbb{R}_+ \)
- \( \mathbb{E}(\log U) = -\psi_0(\theta) \) and \( \text{Var}(\log U) = \psi_1(\theta) \)
- \( \psi_n(s) = \left( d^{n+1} / ds^{n+1} \right) \log \Gamma(s) \)
Variance bounds for log $Z$

With $0 < \theta < \mu$ fixed and $N \nearrow \infty$ assume

$$|m - N\Psi_1(\mu - \theta)| \leq CN^{2/3} \quad \text{and} \quad |n - N\Psi_1(\theta)| \leq CN^{2/3} \quad (1)$$
Variance bounds for log $Z$ 

With $0 < \theta < \mu$ fixed and $N \uparrow \infty$ assume 

$$|m - N\Psi_1(\mu - \theta)| \leq CN^{2/3} \quad \text{and} \quad |n - N\Psi_1(\theta)| \leq CN^{2/3}$$  \hspace{1cm} (1) 

**Theorem** 

For $(m, n)$ as in (1), 

$$C_1 N^{2/3} \leq \text{Var}(\log Z_{m,n}) \leq C_2 N^{2/3}.$$ 

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Outline: Introduction KPZ equation Log-gamma polymer
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Theorem

For $(m, n)$ as in (1), $C_1 N^{2/3} \leq \text{Var}(\log Z_{m,n}) \leq C_2 N^{2/3}$.

Theorem

Suppose $n = \Psi_1(\theta)N$ and $m = \Psi_1(\mu - \theta)N + \gamma N^{\alpha}$ with $\gamma > 0$, $\alpha > 2/3$. Then

$$N^{-\alpha/2} \left\{ \log Z_{m,n} - \mathbb{E}(\log Z_{m,n}) \right\} \Rightarrow \mathcal{N}(0, \gamma \Psi_1(\theta))$$
Fluctuation bounds for path

\( v_0(j) = \text{leftmost, } v_1(j) = \text{rightmost point of } x. \) on horizontal line:

\[
\begin{align*}
v_0(j) &= \min\{i \in \{0, \ldots, m\} : \exists k : x_k = (i, j)\} \\
v_1(j) &= \max\{i \in \{0, \ldots, m\} : \exists k : x_k = (i, j)\}
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**Theorem**

Assume \((m, n)\) as previously and \(0 < \tau < 1\). Then

(a) \( P\left\{ v_0(\lfloor \tau n \rfloor) < \tau m - bN^{2/3} \quad \text{or} \quad v_1(\lfloor \tau n \rfloor) > \tau m + bN^{2/3} \right\} \leq \frac{C}{b^3} \)

(b) \( \forall \varepsilon > 0 \exists \delta > 0 \text{ such that} \)

\[
\lim_{N \to \infty} P\left\{ \exists k \text{ such that } |x_k - (\tau m, \tau n)| \leq \delta N^{2/3} \right\} \leq \varepsilon.
\]
Results for log-gamma polymer summarized

With reciprocals of gammas for weights, both endpoints of the polymer fixed and the right boundary conditions on the axes, we have identified the one-dimensional exponents

\[ \zeta = \frac{2}{3} \quad \text{and} \quad \chi = \frac{1}{3}. \]
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Next step is to
- eliminate the boundary conditions and
- consider polymers with fixed length and free endpoint.
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In both scenarios we have the upper bounds for both log \( Z \) and the path.

But currently do not have the lower bounds.
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Next some key points of the proof.
Burke property for log-gamma polymer with boundary

Given initial weights \((i, j \in \mathbb{N})\):

\[
U^{-1}_{i,0} \sim \text{Gamma}(\theta), \quad V^{-1}_{0,j} \sim \text{Gamma}(\mu - \theta) \quad Y^{-1}_{i,j} \sim \text{Gamma}(\mu)
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Y_{i,j}^{-1} \sim \text{Gamma}(\mu)
\]

Compute \(Z_{m,n}\) for all \((m, n) \in \mathbb{Z}_+^2\) and then define

\[
U_{m,n} = \frac{Z_{m,n}}{Z_{m-1,n}} \quad V_{m,n} = \frac{Z_{m,n}}{Z_{m,n-1}} \quad X_{m,n} = \left(\frac{Z_{m,n}}{Z_{m+1,n}} + \frac{Z_{m,n}}{Z_{m,n+1}}\right)^{-1}
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Y_{i,j}^{-1} \sim \text{Gamma}(\mu)
\]

Compute \(Z_{m,n}\) for all \((m, n) \in \mathbb{Z}_+^2\) and then define

\[
U_{m,n} = \frac{Z_{m,n}}{Z_{m-1,n}} \quad V_{m,n} = \frac{Z_{m,n}}{Z_{m,n-1}} \quad X_{m,n} = \left( \frac{Z_{m,n}}{Z_{m+1,n}} + \frac{Z_{m,n}}{Z_{m,n+1}} \right)^{-1}
\]

For an undirected edge \(f\):

\[
T_f = \begin{cases} 
U_x & f = \{x - e_1, x\} \\
V_x & f = \{x - e_2, x\}
\end{cases}
\]
down-right path \((z_k)\) with edges \(f_k = \{z_{k-1}, z_k\}, \ k \in \mathbb{Z}\)

- interior points \(I\) of path \((z_k)\)
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**Theorem**

Variables \(\{T_{f_k}, X_z : k \in \mathbb{Z}, \ z \in I\}\) are independent with marginals
\(U^{-1} \sim \text{Gamma}(\theta), \ V^{-1} \sim \text{Gamma}(\mu - \theta),\)
and \(X^{-1} \sim \text{Gamma}(\mu).\)
**Theorem**

Variables \( \{ T_{f_k}, X_z : k \in \mathbb{Z}, z \in \mathcal{I} \} \) are independent with marginals

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U^{-1} \sim \text{Gamma}(\theta), \quad V^{-1} \sim \text{Gamma}(\mu - \theta),
\]

and \( X^{-1} \sim \text{Gamma}(\mu) \).

“Burke property” because the analogous property for last-passage is a generalization of Burke’s Theorem for M/M/1 queues, via the last-passage representation of M/M/1 queues in series.
Proof of Burke property

Induction on \( I \) by flipping a growth corner:

\[
U' = Y(1 + U/V) \quad V' = Y(1 + V/U)
\]

\[
X = (U^{-1} + V^{-1})^{-1}
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Lemma. Given that $(U, V, Y)$ are independent positive r.v.’s, $(U', V', X) \overset{d}{=} (U, V, Y)$ iff $(U, V, Y)$ have the gamma distr’s.

Proof. “if” part by computation, “only if” part from a characterization of gamma due to Lukacs (1955). □
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$V \bullet \ Y \bullet \ U \quad U' = Y(1 + U/V) \quad V' = Y(1 + V/U) \quad X = (U^{-1} + V^{-1})^{-1}$

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Proof. “if” part by computation, “only if” part from a characterization of gamma due to Lukacs (1955). □

This gives all $(z_k)$ with finite $I$. General case follows.
Proof of off-characteristic CLT

Recall that

\[
\begin{align*}
    n &= \Psi_1(\theta)N \\
    m &= \Psi_1(\mu - \theta)N + \gamma N^\alpha
\end{align*}
\]

\(\gamma > 0\), \(\alpha > 2/3\).
Proof of off-characteristic CLT

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\[
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\]

Set \(m_1 = \lfloor \Psi_1(\mu - \theta)N \rfloor\).
Proof of off-characteristic CLT

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Proof of off-characteristic CLT

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Set \(m_1 = \lfloor \Psi_1(\mu - \theta)N \rfloor\). Since \(Z_{m,n} = Z_{m_1,n} \cdot \prod_{i=m_1+1}^{m} U_{i,n}\)

\[
N^{-\alpha/2} \log Z_{m,n} = N^{-\alpha/2} \log Z_{m_1,n} + N^{-\alpha/2} \sum_{i=m_1+1}^{m} \log U_{i,n}
\]

First term on the right is \(O(N^{1/3 - \alpha/2}) \to 0\). Second term is a sum of order \(N^\alpha\) i.i.d. terms. \(\Box\)
Exit point of path from $x$-axis

$$\xi_x = \max\{k \geq 0 : x_i = (i, 0) \text{ for } 0 \leq i \leq k\}$$
Variance identity

Exit point of path from $x$-axis

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For $\theta, x > 0$ define positive function

$$L(\theta, x) = \int_0^x (\Psi_0(\theta) - \log y) x^{-\theta} y^{\theta-1} e^{x-y} \, dy$$
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**Theorem.** For the model with boundary,

$$\text{Var}[\log Z_{m,n}] = n\Psi_1(\mu - \theta) - m\Psi_1(\theta) + 2 E_{m,n} \left[ \sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1}) \right]$$
Variance identity, sketch of proof

\[ W = \log Z_{0,n} \]
\[ N = \log Z_{m,n} - \log Z_{0,n} \]
\[ S = \log Z_{m,0} \]
\[ E = \log Z_{m,n} - \log Z_{m,0} \]
Variance identity, sketch of proof

\[ N = \log Z_{m,n} - \log Z_{0,n} \]

\[ W = \log Z_{0,n} \]

\[ E = \log Z_{m,n} - \log Z_{m,0} \]

\[ S = \log Z_{m,0} \]

\[ \text{Var}[\log Z_{m,n}] = \text{Var}(W + N) \]

\[ = \text{Var}(W) + \text{Var}(N) + 2 \text{Cov}(W, N) \]

\[ = \text{Var}(W) + \text{Var}(N) + 2 \text{Cov}(S + E - N, N) \]

\[ = \text{Var}(W) - \text{Var}(N) + 2 \text{Cov}(S, N) \quad (E, N \text{ ind.}) \]

\[ = n\Psi_1(\mu - \theta) - m\Psi_1(\theta) + 2 \text{Cov}(S, N). \]
To differentiate w.r.t. parameter $\theta$ of $S$ while keeping $W$ fixed, introduce a separate parameter $\rho (= \mu - \theta)$ for $W$.

$$-\text{Cov}(S, N) = \frac{\partial}{\partial \theta} \mathbb{E}(N)$$
To differentiate w.r.t. parameter $\theta$ of $S$ while keeping $W$ fixed, introduce a separate parameter $\rho (= \mu - \theta)$ for $W$.

$$-\text{Cov}(S, N) = \frac{\partial}{\partial \theta} \mathbb{E}(N) = \tilde{\mathbb{E}} \left[ \frac{\partial}{\partial \theta} \log Z_{m,n}(\theta) \right]$$
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when $Z_{m,n}(\theta) = \sum_{x, \in \Pi_{m,n}} \prod_{i=1}^{\xi_x} H_\theta(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k}$ with

$$\eta_i \sim \text{IID Unif}(0, 1), \quad H_\theta(\eta) = F_\theta^{-1}(\eta), \quad F_\theta(x) = \int_0^x \frac{y^{\theta-1} e^{-y}}{\Gamma(\theta)} \, dy.$$
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Differentiate: 

$$\frac{\partial}{\partial \theta} \log Z_{m,n}(\theta) = - E^{Q_{m,n}} \left[ \sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1}) \right].$$
Together:

$$\text{Var}[\log Z_{m,n}] = n\Psi_1(\mu - \theta) - m\Psi_1(\theta) + 2 \text{Cov}(S, N)$$

$$= n\Psi_1(\mu - \theta) - m\Psi_1(\theta) + 2 E_{m,n} \left[ \sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1}) \right].$$

This was the claimed formula. $\square$
Sketch of upper bound proof

The argument develops an inequality that controls both $\log Z$ and $\xi_x$ simultaneously. Introduce an auxiliary parameter $\lambda = \theta - bu/N$. 

$W(\theta) = \xi_x \prod_{i=1}^N H_{\theta}(\eta_i) - 1 \cdot m^n \prod_{k=1}^{\xi_x+1} Y_{x_k} W(\lambda) \cdot \xi_x \prod_{i=1}^N H_{\lambda}(\eta_i) H_{\theta}(\eta_i)$.

$Q_{\theta,\omega}\{\xi_x \geq u\} = Z(\theta) \sum_x 1 \{\xi_x \geq u\} W(\theta) \leq Z(\lambda) Z(\theta) \cdot \lfloor u \rfloor \prod_{i=1}^N H_{\lambda}(\eta_i) H_{\theta}(\eta_i)$. 

Scaling for a polymer
The argument develops an inequality that controls both log $Z$ and $\xi_x$ simultaneously. Introduce an auxiliary parameter $\lambda = \theta - bu/N$. The weight of a path $x$, such that $\xi_x > 0$ satisfies

$$W(\theta) = \prod_{i=1}^{\xi_x} H_\theta(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k}$$
The argument develops an inequality that controls both log $Z$ and $\xi_x$ simultaneously. Introduce an auxiliary parameter $\lambda = \theta - bu/N$. The weight of a path $x$, such that $\xi_x > 0$ satisfies

$$W(\theta) = \prod_{i=1}^{\xi_x} H_\theta(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k} = W(\lambda) \cdot \prod_{i=1}^{\xi_x} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)}.$$
Sketch of upper bound proof

The argument develops an inequality that controls both $\log Z$ and $\xi_x$ simultaneously. Introduce an auxiliary parameter $\lambda = \theta - bu/N$. The weight of a path $x$, such that $\xi_x > 0$ satisfies

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Since $H_\lambda(\eta) \leq H_\theta(\eta)$,

$$Q^{\theta,\omega} \{ \xi_x \geq u \} = \frac{1}{Z(\theta)} \sum_x 1\{ \xi_x \geq u \} W(\theta) \leq \frac{Z(\lambda)}{Z(\theta)} \cdot \prod_{i=1}^{\lfloor u \rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)}.$$
For $1 \leq u \leq \delta N$ and $0 < s < \delta$, 

\[
\mathbb{P}
\left[
\mathcal{Q}^\omega \{ \xi_x \geq u \} \geq e^{-su^2/N}
\right]
\leq \mathbb{P}\left\{ \prod_{i=1}^{[u]} \frac{H_{\lambda}(\eta_i)}{H_{\theta}(\eta_i)} \geq \alpha \right\}
\]

\[
+ \mathbb{P}\left( \frac{Z(\lambda)}{Z(\theta)} \geq \alpha^{-1} e^{-su^2/N} \right).
\]
For $1 \leq u \leq \delta N$ and $0 < s < \delta$,

\[
\mathbb{P}\left[ Q^\omega \{ \xi_x \geq u \} \geq e^{-su^2/N} \right] \leq \mathbb{P}\left\{ \prod_{i=1}^{[u]} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)} \geq \alpha \right\}
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+ \mathbb{P}\left( \frac{Z(\lambda)}{Z(\theta)} \geq \alpha^{-1} e^{-su^2/N} \right).
\]

Choose $\alpha$ right. Bound these probabilities with Chebychev which brings $\text{Var}(\log Z)$ into play. In the characteristic rectangle $\text{Var}(\log Z)$ can be bounded by $E(\xi_x)$. The end result is this inequality:

\[
\mathbb{P}\left[ Q^\omega \{ \xi_x \geq u \} \geq e^{-su^2/N} \right] \leq \frac{CN^2}{u^4} E(\xi_x) + \frac{CN^2}{u^3}
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For $1 \leq u \leq \delta N$ and $0 < s < \delta$,

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Handle $u \geq \delta N$ with large deviation estimates. In the end, integration gives the moment bounds.
For $1 \leq u \leq \delta N$ and $0 < s < \delta$,

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$$+ \mathbb{P}\left( \frac{Z(\lambda)}{Z(\theta)} \geq \alpha^{-1} e^{-su^2/N} \right).$$

Choose $\alpha$ right. Bound these probabilities with Chebychev which brings $\mathbb{V}ar(\log Z)$ into play. In the characteristic rectangle $\mathbb{V}ar(\log Z)$ can be bounded by $E(\xi_x)$. The end result is this inequality:

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Handle $u \geq \delta N$ with large deviation estimates. In the end, integration gives the moment bounds. **END.**