FLUCTUATION RESULTS FOR SOME MODELS OF RANDOM ENVIRONMENTS AND INTERACTION

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I. RANDOM WALK IN RANDOM ENVIRONMENT

- **Functional central limit theorem for ballistic walk**

II. ONE-DIMENSIONAL INTERACTING SYSTEMS

- **Random average process:** $n^{1/4}$ fluctuations
- **Asymmetric simple exclusion process:**
  $n^{1/3}$ fluctuations

PART I RWRE on $\mathbb{Z}^d$

RWRE is a Markov chain X_n on $\mathbf{Z}^d$ with transition matrix created by a spatially ergodic random mechanism.

Attach to each point $x \in \mathbf{Z}^d$ a random, independently chosen, probability vector $\omega_x = (\omega_{x,u})_{u \in \mathbf{Z}^d}$.

$\sum_u \omega_{x,u} = 1$, $\omega_{x,u}$ = probability of jump u out of x.

$\omega = (\omega_x)_{x \in \mathbf{Z}^d}$ is an environment.

$\mathbb{P}$ is the probability measure on environments.

Standing assumption: vectors ω_x IID under $\mathbb{P}$

Transition probabilities are $\pi_{x,y}(\omega) = \omega_{x,\,y-x}$.

Given ω and initial state z, probability measure P_z^ω of the random walk X_n determined by

$$P_z^\omega(X_0 = z) = 1$$

$$P_z^\omega(\,X_{n+1} = y \,|\, X_n = x\,) = \pi_{x,y}(\omega)$$

P_z^ω is the *quenched* path distribution
(ω fixed but walk X_n random).

$P_z = P_z^\omega(\cdot)\,\mathbb{P}(d\omega)$ is the *averaged (annealed)* distribution
(both ω and walk X_n random).

Usual assumptions: Nearest-neighbor jumps and uniform ellipticity:

$$\begin{cases} \pi_{x,y}(\omega) \ge \kappa > 0, & |x-y| = 1 \\ \pi_{x,y}(\omega) = 0, & |x-y| \ne 1. \end{cases}$$

BASIC QUESTIONS: recurrence/transience, LLN, CLT

Dimension $d = 1$ fairly well understood, but $d \geq 2$ not.

Solomon (1975): recurrence/transience criteria and law of large numbers for $d = 1$.

No general recurrence/transience criteria for $d \geq 2$. Special cases with additional symmetry have been treated.

Law of large numbers: for $d = 2$ known that

$$X_n/n \longrightarrow \text{constant limit}$$

(Sznitman, Zerner) Not known for $d \geq 3$.

Central limit theorems proved under specialized assumptions.

FUNCTIONAL CENTRAL LIMIT THEOREMS

Assuming that a limit velocity $v = \lim X_n/n$ exists, let

$$B_n(t) = n^{-1/2}\{X_{[nt]} - ntv\} \qquad (t \geq 0)$$

Let W be the distribution of a Brownian motion on $\mathbf{R}^d$ with some diffusion matrix Γ. There are two kinds of CLT's:

- Averaged: $P_0\{B_n \in \cdot\} \longrightarrow W(\cdot)$
- Quenched: $P_0^\omega\{B_n \in \cdot\} \longrightarrow W(\cdot)$ for $\mathbb{P}$-a.e. ω

(Quenched) $\Longrightarrow$ (Averaged) by integrating out ω.

THE BALLISTIC CASE

Ballistic walks are those that satisfy $X_n/n \to v \neq 0$.

We make this assumption via regeneration times.

Assume directional transience:

$$\exists \hat{u} \in \mathbf{Z}^d \ \text{ s.t. } \ P_0\{X_n \cdot \hat{u} \to \infty\} = 1.$$

Then w.p. 1 there exists first time τ_1 such that

$$\sup_{n<\tau_1} X_n \cdot \hat{u} \ < \ X_{\tau_1} \cdot \hat{u} \ \le \ \inf_{n \ge \tau_1} X_n \cdot \hat{u}$$

Iteration gives sequence $\tau_1 < \tau_2 < \tau_3 < \cdots < \infty$

(Kalikow; Kesten; Sznitman and Zerner)

Not stopping times, yet $\{X_{\tau_k} - X_{\tau_{k-1}}\}_{k\geq 2}$ IID under P_0.

Define backtracking time $\beta = \inf\{n : X_n \cdot \hat{u} < X_0 \cdot \hat{u}\}$.

From classic IID results and some estimation follow:

- If $E_0(\tau_1) < \infty$ then a.s. $X_n/n \to v \equiv \dfrac{E_0[X_{\tau_1} \,|\, \beta = \infty]}{E_0[\tau_1 \,|\, \beta = \infty]}$

- If $E_0(\tau_1^2) < \infty$ then averaged CLT holds. Limiting Brownian motion has diffusion matrix

$$\Gamma = \frac{E_0\Big[(X_{\tau_1} - \tau_1 v)(X_{\tau_1} - \tau_1 v)^t \,\Big|\, \beta = \infty\Big]}{E_0[\tau_1 \,|\, \beta = \infty]}$$

NATURAL STRATEGY FOR QUENCHED CLT:

Use averaged CLT and control the difference

$$E_0^\omega[F(B_n(\cdot))] - E_0[F(B_n(\cdot))]$$

for rich enough class of $\{F\}$ on path space.

THEOREM (Bolthausen and Sznitman, 2002) Assume $d \geq 4$, non-nestling, and small noise. Then quenched CLT holds.

Non-nestling means $\sum_z (z \cdot \hat{u})\, \pi_{0,z}(\omega) \geq \eta > 0$ $\mathbb{P}$-a.s.

THEOREM (Berger and Zeitouni, 2007) Assume $d \geq 4$, the averaged CLT, and some (low) moments on τ_1 and $\tau_2 - \tau_1$. Then quenched CLT holds.

A QUENCHED CLT FOR BALLISTIC WALKS FOR DIMENSIONS $d \geq 2$

Assumptions

- Directional transience $X_n \cdot \hat{u} \to \infty$
- $E_0(\tau_1^p) < \infty$ for a large p (e.g. $p > 176d$ suffices)
- Bounded steps (not necessarily nearest neighbor)
- Instead of ellipticity, assume that walk not restricted to a 1-dimensional subspace, and

$$\mathbb{P}\{\, \exists\, z \,:\, \pi_{0,0} + \pi_{0,z} = 1 \,\} < 1.$$

THEOREM (Rassoul-Agha, S. 2007) Under these assumptions quenched CLT holds.

Note The regularity assumption is the right one.

MOMENT ASSUMPTION FOR τ_1 GUARANTEED BY ANY ONE OF THESE:

- Non-nestling: $\mathbb{P}\Big\{\omega \,:\, \sum_z (z \cdot \hat{u})\, \pi_{0,z}(\omega) \geq \eta > 0\Big\} \;=\; 1$

- In the uniformly elliptic nearest neighbor case,

$$\mathbb{E}\Big[\Big(\sum_z z \cdot \hat{u}\; \pi_{0,z}\Big)^{+}\Big] \;>\; \kappa^{-1}\; \mathbb{E}\Big[\Big(\sum_z z \cdot \hat{u}\; \pi_{0,z}\Big)^{-}\Big].$$

- Sznitman's (T'): $E_0\Big[\exp(c \sup_{0\leq n\leq \tau_1} |X_n|^{\gamma})\Big] < \infty,\;\; 0 < \gamma < 1$

Believed that in uniformly elliptic case, (T') $\Longleftrightarrow$ ballistic

Not proved, but if we accept it then this last quenched CLT covers all ballistic uniformly elliptic walks.

ABOUT THE PROOF:

Approach based on showing quenched mean subdiffusive:

$$\mathbb{E}(\,|E_0^\omega(X_n) - E_0(X_n)|^2\,) \le Cn^{1-\delta}$$

From this we derive a martingale decomposition

$$X_n - nv = M_n + R_n$$

with error R_n that can be controlled.

This fails in one dimension. More generally, if the earlier regularity assumption fails (while other assumptions are met), $E_0^\omega(X_n)$ itself satisfies a CLT. $\{B_n\}$ **not tight**, but there is a quenched CLT for

$$\widetilde{B}_n(t) = n^{-1/2}\{X_{[nt]} - E_0^\omega(X_{[nt]})\}$$

Note that averaged CLT for B_n does hold.

SPECIAL CASE: SPACE-TIME RWRE

Suppose $\pi_{0,z} > 0$ only if $e_1 \cdot z = 1$ so $(X_n - X_{n-1}) \cdot e_1 = 1$.

In other words, e_1 = "time direction" $(d \geq 2)$.

Averaged walk is a classical IID random walk.

For this case we have the ideal quenched CLT:

THEOREM (Rassoul-Agha, S. 2005) Under assumptions

$$\sum_z |z|^2 \, \mathbb{E}(\pi_{0,z}) < \infty \, , \; \mathbb{P}(\sup_z \pi_{0,z} = 1) < 1$$

quenched invariance principle holds: under P_0^ω,

$$B_n(t) \; = \; n^{-1/2}\{X_{[nt]} - ntv\} \; \longrightarrow \; B(t)$$

where B is Brownian motion with diffusion matrix

$$\Gamma = \sum_z (z-v)(z-v)^T \mathbb{E}(\pi_{0,z})$$

SPACE-TIME RWRE

Marginal quenched CLT proved in several stages under progressively more general assumptions:

Boldrighini, Minlos, and Pellegrinotti (PTRF 1997) under small noise, $d \geq 3$;
Bernabei, Boldrighini, Minlos, and Pellegrinotti (MPRF 1998) under small noise, $d \geq 2$;
Boldrighini, Minlos, and Pellegrinotti (PTRF 2004) under uniform exponential tail bounds on jump kernel
Stannat (PTRF 2004) simplified proof of B-B-M-P 1998 result
Bérard (JAP 2004) elementary proof for $d = 2$, nearest neighbor, averaged walk symmetric

PART II INTERACTING SYSTEMS

1. RANDOM AVERAGE PROCESS (RAP)

A simple interacting random system on $\mathbf{Z}$,
original motivation for study of the space-time RWRE.

State of the process: height function $\sigma : \mathbf{Z} \to \mathbf{R}$

Time evolution: discrete time process σ_τ, $\tau = 0, 1, 2, \ldots,$

$$\sigma_\tau(k) = \sum_j u_j(k,\tau)\sigma_{\tau-1}(k+j)\,, \qquad k \in \mathbf{Z}$$

$(u_j(k,\tau) : -M \le j \le M)$ is a random probability vector, IID over space-time points (k,τ).

Assumption: Weights not degenerate.

SETTING FOR LIMITS

- sequence of processes σ^n_τ $\;(n = 1, 2, 3, \dots)$
- $\sigma^n_0(0) = 0$
- independent initial increments $\eta^n_0(k) = \sigma^n_0(k) - \sigma^n_0(k-1)$

with

$$\mathbf{E}\,[\eta^n_0(k)] = \rho(k/n)\,, \qquad \mathbf{Var}\,[\eta^n_0(k)] = v(k/n)$$

for Hölder $1/2+\varepsilon$ functions ρ and v, plus uniform $2+\varepsilon$ moment bound.

Let $U(x) \;=\; \int_0^x \rho(y)\,dy$ and $b = -\sum_j j\,\mathbb{E}u_j(k,\tau)$

LLN: $n^{-1}\,\sigma^n_{[nt]}([nx]) \longrightarrow U(x-bt)$ (prob)

FLUCTUATIONS AROUND A CHARACTERISTIC

To follow evolution around characteristic $t \mapsto \bar{y} + tb$ emanating from fixed $\bar{y} \in \mathbf{R}$, define space-time process

$$Z_n(t,r) = \sigma^n_{[nt]}([n\bar{y}] + [r\sqrt{n}\,] + [ntb]) \;-\; \sigma^n_0([n\bar{y}] + [r\sqrt{n}\,])$$

indexed by $(t,r) \in \mathbf{R}_+ \times \mathbf{R}$.

THEOREM [Balázs, Rassoul-Agha, S. 2006]

Weak limit $n^{-1/4} Z_n \Rightarrow z$ in the sense of convergence of finite-dimensional distributions.

Limit process $\{z(t,r) : t \geq 0,\, r \in \mathbf{R}\}$ is mean zero Gaussian and satisfies $\{z(at, \sqrt{a}r)\} \overset{d}{=} \{a^{1/4} z(t,r)\}$.

LIMIT PROCESS z

$$
\begin{aligned}
z(t,r) \;=\; & \sigma_a \rho(\bar{y})\sqrt{\kappa} \iint\limits_{[0,t]\times\mathbf{R}} \varphi_{\sigma_a^2(t-s)}(r-z)\, dW(s,z) \\
& + \;\sqrt{v(\bar{y})} \int\limits_{\mathbf{R}} \mathrm{sign}(x-r)\, \Phi_{\sigma_a^2 t}\Big(-|x-r|\Big)\, dB(x)
\end{aligned}
$$

W = 2-parameter Brownian motion on $\mathbf{R}_+ \times \mathbf{R}$

B = 1-parameter Brownian motion on $\mathbf{R}$

W and B independent

$$
\varphi_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big\{-\frac{x^2}{2\sigma^2}\Big\}, \quad \Phi_{\sigma^2}(x) = \int_{-\infty}^{x} \varphi_{\sigma^2}(y)\, dy.
$$

FRACTIONAL BROWNIAN MOTION

Self-similarity $\{z(at, \sqrt{a}r)\} \overset{d}{=} \{a^{1/4} z(t,r)\}$ suggests fBM with Hurst parameter $H = 1/4$.

This happens if $v(\bar{y}) = \kappa \rho(\bar{y})^2$: then

$$\mathbf{E}\, z(s,r) z(t,r) = c \left(\sqrt{s} + \sqrt{t} - \sqrt{|t-s|}\ \right)$$

which identifies $z(\,\cdot\,, r)$ as fBM.

IID invariant distributions for increments $\{\eta(k)\}$ known in one family of cases.

fBM limit arises in this stationary situation.

PROOF IDEA: BACKWARD WALKS

IID weights define **environment** $\omega = \{u(k,\tau) : \tau, k \in \mathbf{Z}\}$.

RAP evolution again:

$$\begin{aligned}\sigma_\tau(i) = \sum_j u_{j-i}(i,\tau)\sigma_{\tau-1}(j) &= E^\omega\Big[\sigma_{\tau-1}(X_1^{i,\tau})\Big] \\ = \;\cdots\; &= E^\omega\Big[\sigma_0(X_\tau^{i,\tau})\Big]\end{aligned}$$

where $X_k^{i,\tau}$ is a space-time RWRE started from (i,τ).

E^ω is the expectation over the path of $X_k^{i,\tau}$, a new level of randomness.

CALCULATION with $x(n,r) = [n\bar{y}] + [r\sqrt{n}]$:

$$
\begin{aligned}
Z_n(t,r) &= \sigma^n_{[nt]}(x(n,r)+[ntb]) - \sigma^n_0(x(n,r)) \\
&= E^\omega\Big[\sigma^n_0\big(X^{x(n,r)+[ntb]}_{[nt]}\big) - \sigma^n_0(x(n,r))\Big] \\
&= \sum_{i\in\mathbf{Z}} \Big(\eta^n_0(i) - \rho(i/n)\Big)\Big(\mathbf{1}\{i > x(n,r)\}P^\omega\big\{X^*_{[nt]} \ge i\big\} \\
&\qquad\qquad - \mathbf{1}\{i \le x(n,r)\}P^\omega\big\{X^*_{[nt]} < i\big\}\Big) \\
&\qquad + \rho(\bar{y})\, E^\omega\Big(X^*_{[nt]} - x(n,r)\Big) + o(n^{1/4}).
\end{aligned}
$$

Quenched CLT + Lindeberg-Feller handles first sum. Second term needs limit for quenched mean.

TO CONCLUDE, NEED LIMIT FOR BACKWARD QUENCHED MEAN PROCESS

Process

$$y_n(t,r) = n^{-1/4}\Big\{ E^{\omega}\Big(X_{[nt]}^{[ntb]+[r\sqrt{n}\,],[nt]} \Big) - [r\sqrt{n}\,] \Big\}$$

for $(t,r) \in \mathbf{R}_+ \times \mathbf{R}$

THEOREM $y_n(t,r) \Rightarrow y(t,r)$ in the sense that finite-dimensional distributions converge. Limit y is weak solution of a stochastic heat equation:

$$y_t = \tfrac{1}{2}\sigma_a^2\, y_{rr} + c\dot{W} \qquad y(0,r) \equiv 0.$$

This result and quenched CLT for RWRE plus Lindeberg-Feller combine to prove limit for RAP.

PART II.2 ASYMMETRIC SIMPLE EXCLUSION PROCESS (ASEP)

State of the process: height function $h : \mathbf{Z} \to \mathbf{Z}$ s.t. $h_i - h_{i-1} = 0$ or 1

Continuous time evolution: Height variables h_i jump **down** with rate p, **up** with rate $q = 1 - p$, but forced to preserve $0 \le h_i - h_{i-1} \le 1$.

Consider stationary ASEP with mean ρ Bernoulli increments $\eta_i = h_i - h_{i-1}$. Normalize $h_0(0) = 0$.

Let $V^\rho = (p - q)(1 - 2\rho)$ ("characteristic speed").

THEOREM [P.L.Ferrari and Spohn, 2005] For stationary TASEP ($p = 1$, $q = 0$)

$$\lim_{t\to\infty} \mathbf{P}\left\{ \frac{h_{[V^\rho t]}(t) - \mathbf{E}\left[h_{[V^\rho t]}(t)\right]}{c\, t^{1/3}} \le s \right\} = F(s)$$

F related to Tracy-Widom distributions from random matrix theory.

Builds on seminal work of Baik, Deift and Johansson (1999), depends on underlying determinantal structure.

General ASEP apparently not determinantal.

THEOREM [Balázs and S. 2006] For general $0 < p = 1 - q \le 1$ with $p \ne 1/2$, for $t \ge 1$,

$$C_1\, t^{2/3} \le \mathbf{Var}\left\{h_{[V^\rho t]}(t)\right\} \le C_2\, t^{2/3}$$

COMMENTS

- **Characteristic speed.** On larger space and time scales height process obeys the PDE

$$u_t + f(u_x) = 0 \qquad \text{with} \quad f(\rho) = (p-q)\rho(1-\rho)$$

This is made rigorous via a "hydrodynamic limit"

$$n^{-1} h^n_{[nx]}(nt) \longrightarrow u(t,x)$$

Characteristics of the PDE are solutions of $\dot{x} = f'(u_x(t,x))$.

At constant density (slope) ρ the speed is

$$f'(\rho) = V^\rho = (p-q)(1-2\rho)$$

At $V \neq V^\rho$ height $h_{[Vt]}(t)$ sees translated initial Gaussian fluctuations on scale $t^{1/2}$. (Ferrari and Fontes 1994)

- **Proof idea**

Begins with

$$\textbf{Var}\,\{h_{[Vt]}(t)\} = \rho(1-\rho)\mathbf{E}\Big\{\,|Q(t)-[Vt]\,|\,\Big\}$$

$Q(t)$ is the location of a second-class particle.

This is a disturbance introduced into the process. Macroscopically it tracks the characteristic.

Through coupling of several processes with different initial conditions, evolution of the second class particle can be related to differences in particle current (=height) between the processes.

TWO CLASSES OF FLUCTUATIONS IN ASYMMETRIC SYSTEMS IN ONE DIMENSION

$n^{1/4}$ scaling, Gaussian limit related to fBM with $H = 1/4$	independent walks random average process (flux $f''(\rho) = 0$)
$n^{1/3}$ scaling, Tracy-Widom limit	asymmetric exclusion Hammersley process polynuclear growth model (flux $f''(\rho) \neq 0$)

CURRENT PROJECTS

Adapt techniques introduced for RWRE central limit theorems to prove similar results for tagged particles in interacting particle systems (open for AEP in dim $d = 1, 2$).

Prove $n^{1/3}$ fluctuations for other systems such as zero range process, to support universality.

KPZ picture from stat phys asserts that nonlinearities of order > 2 have no influence on large scale fluctuations, but logarithmic corrections may arise. Attempt to check this with exclusion systems with flux such that $f''(\rho) = 0$.